A SYMMETRIC p-ADIC SYMBOL FOR TRIPLES OF MODULAR FORMS

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Abstract. In [DR14], the authors define the Garrett-Rankin triple product p-adic L-function and relate it to the image of certain diagonal cycles under the p-adic Abel-Jacobi map. We introduce a new p-adic triple symbol based on this p-adic L-function and show that it satisfies symmetry relations, when permuting the three inputted modular forms. We also provide computational evidence confirming that it is indeed cyclic when the modular forms have even weights, and provide counter-examples in the case containing odd weights. To do so, we extend the algorithm provided in [Lau14] to allow for ordinary projections of nearly overconvergent modular forms – not just overconvergent modular forms – as well as certain projections over spaces of non-zero slope. Finally, a curious consequence of our work is an efficient method to calculate certain Poincaré pairings in higher weight.

Contents

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1. Introduction

Let f, g, h be three cuspidal eigenforms over $\mathbb{Q}$ of weight 2, level N and trivial characters. Fix a prime $p \geq 5$ and assume that $p \nmid N$. Let α_f, β_f be the roots of the *Hecke polynomial*

$$x^2 - a_p(f)x + p.$$

Assume that the modular form f is *regular at* p, i.e. that α_f and β_f are different. Assume as well that f is *ordinary at* p, i.e. that one of the roots of $x^2 - a_p(f)x + p$, say α_f, is a p-adic unit. Define the following two modular forms:

$$\begin{aligned} f_\alpha(q) &:= f(q) - \beta_f f(q^p); \\ f_\beta(q) &:= f(q) - \alpha_f f(q^p). \end{aligned} \tag{1}$$

We call f_α and f_β the *p-stabilizations* of f. They have level pN, and are eigenforms for the U_p operator with respective eigenvalues α_f and β_f. Since we assumed that α_f is a unit, it is customary to call f_α the *ordinary p-stabilization* of f. Define the following Euler factors:

$$\begin{gathered} \mathcal{E}(f,g,h) := (1-\beta_f\alpha_g\alpha_h p^{-2})(1-\beta_f\alpha_g\beta_h p^{-2})(1-\beta_f\beta_g\alpha_h p^{-2})(1-\beta_f\beta_g\beta_h p^{-2}); \\ \tilde{\mathcal{E}}(f,g,h) := (1-\alpha_f\alpha_g\alpha_h p^{-2})(1-\alpha_f\alpha_g\beta_h p^{-2})(1-\alpha_f\beta_g\alpha_h p^{-2})(1-\alpha_f\beta_g\beta_h p^{-2}); \\ \mathcal{E}_0(f) := 1-\beta_f^2 p^{-1}; \qquad \tilde{\mathcal{E}}_0(f) := 1-\alpha_f^2 p^{-1}; \\ \mathcal{E}_1(f) := 1-\beta_f^2 p^{-2}; \qquad \tilde{\mathcal{E}}_1(f) := 1-\alpha_f^2 p^{-2}. \end{gathered} \tag{2}$$

Let λ_{f_γ} be the projection over f_γ; it is the unique Hecke-equivariant linear functional that factors through the Hecke eigenspace associated to f_γ and is normalized to send f_γ to 1 (cf. Definition 2.7 in [Loe18]). Let $\mathrm{d} := q\frac{\mathrm{d}}{\mathrm{d}q}$ be the Serre differential operator and $\omega_f := f(q)\frac{dq}{q}$ the differential associated to f. Consider the quantity

$$\frac{\langle \omega_f, \phi(\omega_f)\rangle}{p}\left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)}\ \beta_f\ \lambda_{f_\alpha}\left(\mathrm{d}^{-1}(g^{[p]})\times h\right) + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)}\ \alpha_f\ \lambda_{f_\beta}\left(\mathrm{d}^{-1}(g^{[p]})\times h\right)\right), \tag{3}$$

where $\langle\cdot,\cdot\rangle$ is the Poincaré pairing (cf. Theorem 5.2 of [Col95]) and ϕ is the Frobenius map. It turns out that this quantity is independent – up to a sign – of the order of f, g and h. This result is particularly surprising since the quantity in (3) does not appear to be symbolically symmetric in f, g and h. This will fit into the framework of our paper, as we relate this quantity to the image of certain diagonal cycles under the p-adic Abel-Jacobi map.

The above can even be generalized to modular forms of higher weight and any characters satisfying $\chi_f\chi_g\chi_h = 1$, which we will do in Section 4. In that case, one needs to adjust the Euler factors from (2) and introduce an extra factor and some twists by χ_f^{-1} in (3). One would also require that the weights be balanced, i.e. that the largest one is strictly smaller than the sum of the other two.

In order to explicitly calculate (3), for modular forms of general weight, we need certain computational tools, namely being able to compute ordinary projections of nearly overconvergent modular forms, as well as projection over the slope α subspace for α not necessarily zero. In [Lau14] (see also [Lau11]), the author describes an algorithm allowing the calculation of ordinary projections of overconvergent modular forms. We introduce here improvements to this algorithm, allowing us to accomplish the aforementioned tasks. The use of this new algorithm is not restricted to this paper. The experimental calculations detailed in Section

5, on the symmetry of (3), provide additional support to the fact that our algorithm is functioning properly.

An additional application of our code is the calculation of certain periods of modular forms. Indeed, using the symmetry of our new p-adic triple symbol, introduced in Section 4.2, we explain how one can use our algorithms to compute the Poincaré pairing $\Omega_f := \langle \omega_f, \phi(\omega_f)\rangle$, where ϕ denotes the Frobenius action and f is a newform of any weight. See [DL21], [DLR16] and Section III.5 of [Nik11] for instances where this pairing appears in the literature. There are currently no known ways of evaluating general Poincaré pairings, and the value of Ω_f has so far only been computed in cases where f has weight 2 using Kedlaya's algorithm [Ked01].

Roadmap. Our paper is structured as follows. In Section 2, we introduce the main theoretical notions used in this work. That is, the main results concerning overconvergent and nearly overconvergent modular forms, the Katz basis, the U_p operator, and finally Lauder's algorithm introduced in [Lau14] to compute ordinary projections of overconvergent modular forms. In Section 3, we describe improvements to that algorithm. Firstly, we introduce an overconvergent projector, and use it for the calculation of the ordinary projection of a nearly overconvergent modular form. Secondly, we describe a method to compute the projection of a nearly overconvergent modular form over the slope α subspace, where α is not necessarily zero. Section 4 is dedicated to our new p-adic triple symbol. We first describe the Garrett-Rankin triple product p-adic L-function defined in [DR14] and its relation to the p-adic Abel-Jacobi map. We then use this relation to introduce a new p-adic triple symbol $(f, g, h)_p$. After that, we focus on studying the symmetry properties of $(f, g, h)_p$ when permuting f, g and h. Finally, in Section 5, we provide our experimental evidence and show how to compute the Poincaré pairing $\langle \omega_f, \phi(\omega_f)\rangle$, where ϕ denotes the Frobenius action and f is a newform with rational coefficients of any weight.

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2. Background

2.1. **Modular forms.** Throughout this paper, we will mainly deal with overconvergent and nearly overconvergent modular forms, a thorough account of which is given in [Kat73] (for overconvergent forms) and [DR14] or [Urb14] (for their nearly-overconvergent counterpart). In this section, we will review their computational aspects as well as the necessary results that will be used in the rest of the paper.

Let B be a p-adic ring, i.e. a complete separated $\mathbb{Z}_p$-algebra with the p-adic topology, with fraction field K. Denote by $M_k^{p\text{-adic}}(B, \Gamma; r)$ the space of p-adic modular forms of weight k, level Γ and growth condition $r \in B$. We might often drop the B when there is no possible confusion. In the remaining sections, we will have $B = \mathbb{Z}_p$. In the case where r is not a unit, we say that we have an *overconvergent modular form* of weight k, level Γ and growth condition r. We denote the space of such overconvergent modular forms by $M_k^{\mathrm{oc}}(B, \Gamma; r)$.

The usual definitions of overconvergent modular forms, viewing them as functions on test objects or as sections of certain line bundles (cf. [Gou88, Kat73]) are not very amenable to computations, as they are quite abstract. An alternative way to work with such modular forms is through the Katz basis, allowing us to express overconvergent modular forms as series in classical objects.

First, assume that $p \geq 5$ and does not divide N. Let E_{p-1} denote the normalized Eisenstein series of weight $p-1$ (and level 1). We write $M_k(B, N)$ to mean the space of modular forms over B of weight k and level $\Gamma_1(N)$. Note that we have

$$M_k(B, N) = M_k(\mathbb{Z}_p, N) \otimes_{\mathbb{Z}_p} B.$$

The map

$$\begin{aligned} M_{k+(i-1)(p-1)}(B, N) &\hookrightarrow M_{k+i(p-1)}(B, N) \\ f &\mapsto E_{p-1} \cdot f \end{aligned}$$

is injective but not surjective for all $i \geq 1$. It also has a finite free cokernel ([Kat73], Lemma 2.6.1), so it must split. We can then, following Gouvêa's notation in [Gou88], let $A_{k+i(p-1)}(B, N)$ be a free B-module such that

$$M_{k+i(p-1)}(B, N) = E_{p-1} \cdot M_{k+(i-1)(p-1)}(B, N) \oplus A_{k+i(p-1)}(B, N).$$

For $i = 0$, let $A_k(B, N) := M_k(B, N)$. We also have

$$A_{k+i(p-1)}(B, N) = A_{k+i(p-1)}(\mathbb{Z}_p, N) \otimes_{\mathbb{Z}_p} B.$$

We can think of $A_{k+i(p-1)}(B, N)$ as the set of modular forms of weight $k + i(p-1)$ that do not come from smaller weight forms multiplied by E_{p-1}. We notice that we can write

$$M_{k+i(p-1)}(B, N) = \bigoplus_{a=0}^{i} (E_{p-1})^{i-a} \cdot A_{k+a(p-1)}(B, N).$$

We may now give an equivalent definition for the space of r-overconvergent modular forms.

Proposition 2.1. *The space of overconvergent modular forms of weight k, growth condition r and level $\Gamma_1(N)$ is given by*

$$M_k^{oc}(N; r) = \left\{ \sum_{i=0}^{\infty} r^i \frac{b_i}{E_{p-1}^i} : b_i \in A_{k+i(p-1)}(N), \lim_{i \to \infty} b_i = 0 \right\}, \tag{4}$$

where by $\lim_{i\to\infty} b_i = 0$, we mean that the expansion of b_i is more and more divisible by p as i goes to infinity.

Remark 1. If we take r to be invertible in equation (4), we will just get $M_k^{p\text{-adic}}(N; r)$. Moreover, we can also define overconvergent modular forms with a given character χ to be

$$M_k^{\mathrm{oc}}(N, \chi; r) := \left\{ \sum_{i=0}^{\infty} r^i \frac{b_i}{E_{p-1}^i} : b_i \in A_{k+i(p-1)}(N, \chi), \lim_{i \to \infty} b_i = 0 \right\},$$

where $A_{k+i(p-1)}(N, \chi)$ is define analogously to $A_{k+i(p-1)}(N)$ by

$$M_{k+i(p-1)}(B, N, \chi) = E_{p-1} \cdot M_{k+(i-1)(p-1)}(B, N, \chi) \oplus A_{k+i(p-1)}(B, N, \chi).$$

An expansion for $f \in M_k^{p\text{-adic}}(B, N, \chi; r)$ of the form $f = \sum_{i=0}^{\infty} r^i \frac{b_i}{E_{p-1}^i}$ is called a *Katz expansion.*

Remark 2. Some authors (see [Urb14] for instance) use the notation $M_k^{\mathrm{oc}}(B, N, \chi; \alpha)$ to mean $M_k^{\mathrm{oc}}(B, N, \chi; p^\alpha)$. This is because the actual value of r is unimportant; only its p-adic valuation matters. We will also use this notation, but from Section 2.3 onward.

We can also talk about the space of all overconvergent modular forms $M_k^{\mathrm{oc}}(B, N, \chi)$ without specifying the growth condition,

$$M_k^{\mathrm{oc}}(B, N, \chi) := \bigcup_{r \notin B^\times} M_k^{\mathrm{oc}}(B, N, \chi; r).$$

If we were to include units in the above definition, we'd obtain the set of all p-adic modular forms of level $\Gamma_1(N)$, weight k and any growth condition,

$$M_k^{p\text{-adic}}(B, N, \chi) := \bigcup_{r \in B} M_k^{p\text{-adic}}(B, N, \chi; r).$$

If $r = r_0 r_1$, we then have an inclusion

$$\begin{aligned} M_k^{p\text{-adic}}(B, N, \chi; r) &\hookrightarrow M_k^{p\text{-adic}}(B, N, \chi; r_0), \\ \sum_{i=0}^{\infty} r^i \frac{b_i}{E_{p-1}^i} &\mapsto \sum_{i=0}^{\infty} r_0^i \frac{(r_1^i b_i)}{E_{p-1}^i}. \end{aligned} \tag{5}$$

In particular, letting $r_0 = 1$, or any unit, we see that the space of p-adic modular forms of growth condition 1 is equal to the space of p-adic modular forms of any growth condition, i.e. $M_k^{p\text{-adic}}(B, N, \chi; 1) = M_k^{p\text{-adic}}(B, N, \chi)$. We now introduce the Serre differential operator

$$\begin{aligned} q\frac{\mathrm{d}}{\mathrm{d}q} : M_k^{p\text{-adic}}(B, \Gamma_1(N)) &\longrightarrow M_{k+2}^{p\text{-adic}}(B, \Gamma_1(N)) \\ \sum_n a_n q^n &\mapsto \sum_n n a_n q^n. \end{aligned}$$

This operator does not necessarily preserve overconvergence in general. We do however, have the following special case.

Theorem 2.2 (Theorem 2, [CGJ95])**.** *Let $k \geq 1$ and $f \in M_{1-k}^{oc}(B, \Gamma_1(N))$. Then, $\left(q\frac{d}{dq}\right)^k f \in M_{1+k}^{oc}(B, \Gamma_1(N))$.*

We now discuss nearly overconvergent modular forms (see [DR14, Urb14] for more details). Let $M_k^{\text{n-oc}}(B, \Gamma; r; s)$ denote the space of nearly overconvergent modular form of weight k level Γ, growth condition $r \in B$ and order of near overconvergence less or equal to $s \in \mathbb{Z}_{\geq 0}$. When $s = 0$, we retrieve the usual definition of overconvergent modular forms. We have inclusions

$$M_k^{\mathrm{oc}}(B, \Gamma_1(N); r) \subseteq M_k^{\text{n-oc}}(B, \Gamma_1(N); r; s) \subseteq M_k^{p\text{-adic}}(B, \Gamma_1(N))$$

for all r and s. We now give some concrete characterizations of nearly overconvergent modular forms using the Eisenstein series E_2. Recall that E_2 is transcendental over the ring of overconvergent modular forms (cf. [CGJ95]), so

$$M_k^{\mathrm{oc}}(B, \Gamma)(E_2) \cong M_k^{\mathrm{oc}}(B, \Gamma)(X), \tag{6}$$

where X is a free variable.

Proposition 2.3 (Remark 3.2.2 in [Urb14])**.** *Let f be a nearly overconvergent modular form over B of weight k, level $\Gamma_1(N)$ and order less or equal to s. Then there exist overconvergent modular forms $g_0, g_1, ..., g_s$ with $g_i \in M^{oc}_{k-2i}(B, \Gamma_1(N))$ such that*

$$f = g_0 + g_1 E_2 + ... + g_s E_2^s. \tag{7}$$

By Proposition 2.3 and Equation (6), nearly overconvergent modular forms are polynomials in E_2 with overconvergent modular forms as coefficients, so we can view them as elements of $M_k^{\mathrm{oc}}(B, \Gamma_1(N))(X)$. Hence, on top of having a q-expansion in $B[[q]]$ they also have a polynomial q-expansion in $B[[q]][X]$ (of degree less or equal to s, where s is the order of near overconvergence) that comes from Equation (7). Consider the operator δ_k acting on nearly overconvergent modular forms of weight k defined on polynomial q-expansions as

$$(\delta_k f)(q, X) := q\frac{\mathrm{d}}{\mathrm{d}q} f + kXf(q).$$

Then δ_k sends modular forms of weight k to modular forms of weight $k+2$. Define as well the iterated derivate $\delta_k^s := \delta_{k+2s-2} \circ \delta_{k+2s-4} \circ ... \circ \delta_k$.

Proposition 2.4 (Lemma 3.3.4 in [Urb14])**.** *Let f be a nearly overconvergent modular form of weight k and order less or equal to s such that $k > 2s$. Then for each $i = 0, ..., s$, there exists a unique overconvergent modular form h_i of weight $k - 2i$ such that*

$$f = \sum_{i=0}^{s} \delta_{k-2i}^{i}(h_i).$$

Propositions 2.3 and 2.4 allow us to think about nearly overconvergent modular forms as having an overconvergent component in them. We can hence define the overconvergent projection of nearly overconvergent modular forms as

$$\pi_{\mathrm{oc}}\left(\sum_{i=0}^{s} \delta_{k-2i}^{i}(h_i)\right) := h_0.$$

2.2. **The U_p operator.** We now restrict our attention to the space $M_k^{\mathrm{oc}}(\mathbb{Z}_p, N)$ of overconvergent modular forms of level N and weight k. As in the above, we might drop the $\mathbb{Z}_p$ in the notation of M_k^{oc} when the base field is obvious. Consider the Hecke, Atkin and Frobenius operators T_p, U_p and V acting on p-adic modular forms via

$$\begin{aligned}
T_p &: \sum_n a_n q^n \mapsto \sum_n a_{pn} q^n + \chi(p)\ p^{k-1} \sum_n a_n q^{pn}, \\
U_p &: \sum_n a_n q^n \mapsto \sum_n a_{pn} q^n, \\
V &: \sum_n a_n q^n \mapsto \sum_n a_n q^{pn}.
\end{aligned}$$

The Hecke operator T_p acts on modular forms of level N, for $p \nmid N$. The Atkin operator U_p acts on modular forms of level N, for $p | N$. And lastly, the Frobenius operator V takes modular forms of level N to forms of level pN.

We notice that V is a right inverse for U_p and that $VU_p\left(\sum_n a_n q^n\right) = \sum_{p|n} a_n q^n$. In particular, U_p has no left inverse and we write, for a modular form $f := \sum_n a_n q^n$,

$$U_pV(f) = f, \qquad f^{[p]} := (U_pV - VU_p)(f) = \sum_{p\nmid n} a_n q^n.$$

We call $f^{[p]}$ the p-depletion of f. We have the formula $U_p(V(f)\cdot g) = f \cdot U_p(g)$, for modular forms f and g, which can be proven by looking at q-expansions. In particular, this says that U_p is multiplicative when one of its inputs is in the image of the *Frobenius* map V.

Define the p-adic Banach space $M_k^{\mathrm{oc}}(K, N; r) := M_k^{\mathrm{oc}}(B, N; r) \otimes_B K$, where the unit ball is given by $M_k^{\mathrm{oc}}(B, N; r)$. Note that U_p doesn't necessarily preserve the growth conditions of an overconvergent modular form. However, if we restrict our attention to the case $0 < \mathrm{ord}_p(r) < \frac{1}{p+1}$, we have an inclusion

$$\begin{aligned} p\cdot U_p : M_k^{\mathrm{oc}}(B, N; r) &\hookrightarrow M_k^{\mathrm{oc}}(B, N; r^p), \\ f &\mapsto p \cdot U_p(f). \end{aligned}$$

as in Lemma 3.11.4 of [Kat73]. So $U_p\left(M_k^{\mathrm{oc}}(K, N; r)\right) \subseteq \frac{1}{p} M_k^{\mathrm{oc}}(K, N; r^p)$. Combining this with the fact that $M_k^{\mathrm{oc}}(B, N; r^p) \subseteq M_k^{\mathrm{oc}}(B, N; r)$ via the map in (5), we can view the Atkin operator U_p as an endomorphism of $M_k^{\mathrm{oc}}(K, N; r)$ when $0 < \mathrm{ord}_p(r) < \frac{p}{p+1}$. This endomorphism is completely continuous so we can apply p-adic spectral theory, as in [Ser62]. We therefore obtain that the Atkin operator U_p will induce a decomposition (as in Section 2 of [Wan98]), for all $\alpha \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$, on the space of overconvergent modular forms:

$$M_k^{\mathrm{oc}}(K, N; r) = M_k^{\mathrm{oc}}(K, N; r)^{\text{slope } \alpha} \oplus X_\alpha, \tag{8}$$

where $M_k^{\mathrm{oc}}(K, N; r)^{\text{slope } \alpha}$ is the finite dimensional space of overconvergent modular forms in $M_k^{\mathrm{oc}}(K, N; r)$ of slope α. Recall that the slope α subspace is the generalized eigenspace of U_p whose eigenvalues have p-adic valuation α. A similar decomposition to Equation (8) also holds for classical modular forms. If we further assume an infinite slope version of the spectral expansion conjecture (cf. [GM95]), we would obtain

$$M_k^{\mathrm{oc}}(K, N; r) = \widehat{\bigoplus_{\alpha \in \mathbb{Q}_{\geq 0} \cup \{\infty\}}} M_k^{\mathrm{oc}}(K, N; r)^{\text{slope } \alpha}, \tag{9}$$

for $\mathrm{ord}_p(r) \in \left(\frac{1}{p+1}, \frac{p}{p+1}\right)$, where $\widehat{\oplus}$ denotes the completed direct sum. Note that partial results towards the spectral expansion conjecture have been obtained in [Loe07] when $p = 2$.

The overconvergent modular forms of slope zero are said to be ordinary and we denote this space by $M_k^{\mathrm{oc,ord}}(N)$. Actually, Coleman's classicality theorem (cf. [Col95]) states that any ordinary overconvergent modular form of weight $k \geq 2$ can be seen as a classical modular form of weight k on $\Gamma_1(N)$. Therefore, when $k \geq 2$, we can simply denote the ordinary overconvergent modular by $M_k^{\mathrm{ord}}(N)$ instead of $M_k^{\mathrm{oc,ord}}(N)$.

Hida's ordinary projection operator $e_{\mathrm{ord}} := \lim U_{p^{n!}}$ acts on overconvergent modular forms and projects the entire space $M_k^{\mathrm{oc}}(N)$ onto its subspace of ordinary forms $M_k^{\mathrm{oc,\,ord}}(N)$. Finally, Equation (8) tells us that any overconvergent modular form ϕ has a component ϕ_α in each given slope. This gives us a notion of slope projection $e_{\text{slope } \alpha}(\phi) := \phi_\alpha$, with $e_{\text{slope } 0} = e_{\mathrm{ord}}$.

Consider now the space of nearly overconvergent modular forms. One can still define the ordinary projection operator as $e_{\mathrm{ord}} := \lim U_{p^{n!}}$, since this operator is actually defined for any p-adic modular form in general. It turns out that the ordinary projection of a nearly overconvergent modular form only depends on its overconvergent part.

Theorem 2.5 (Lemma 2.7 in [DR14])**.** *Let F be a nearly overconvergent modular form, then*

$$e_{ord}(\phi) = e_{ord}\pi_{oc}(\phi).$$

Thus, taking ordinary projections of *nearly overconvergent* modular forms reduces to taking ordinary projections of *overconvergent* modular forms.

As explained in Section 3.3.6 of [Urb14] (see also Appendix II of [AI21] for Urban's erratum to [Urb14]), the Atkin operator U_p is also completely continuous when viewed as an endomorphism of $M_k^{\text{n-oc}}(K, N; r; s)$, for $0 < \mathrm{ord}_p(r) < \frac{1}{p+1}$. Hence, we obtain a decomposition of $M_k^{\text{n-oc}}(K, N; r; s)$ similar to that of Equation (8). This means that we may also similarly speak of slope α projections $e_{\text{slope }\alpha}(\psi)$ for nearly overconvergent modular forms ψ.

2.3. **Ordinary projections of overconvergent modular forms.** As we are interested in performing explicit computations in this paper, we will approximate our overconvergent modular forms by truncated power series (i.e. polynomials) modulo p^m, in $\mathbb{Z}[[q]]/(q^h, p^m)$ for some $m \in \mathbb{N}$ and $h = h(m, N, k, \chi) \in \mathbb{N}$. Once we know what level of precision we want to obtain after our calculations, we can decide what level of precision we need to start with, as we know how much precision is lost through the algorithms that we use. An alternative – more ad hoc – way to measure the precision of our outputs (p-adic numbers) is to run our algorithm multiple times, to different precisions, and see by what power of p they differ.

We will explain how to write down the Katz expansion of an overconvergent modular form $H \in M_k^{\text{oc}}(\mathbb{Z}_p, N, \chi; \frac{1}{p+1})$ as well as a matrix representing U_p. More details can be found in [Lau14]. Picking a row-reduced basis B_i for each $A_{k+i(p-1)}(\mathbb{Z}_p, N)$, we obtain the sets:

$$\mathrm{Kb} := \left\{ \frac{p^{\left\lfloor \frac{i}{p+1} \right\rfloor} \cdot b}{E_{p-1}^i} \mod (q^{h'p}, p^{m'}) : b \in B_i, i = 1, ..., \left\lfloor \frac{(p+1)m}{p-1} \right\rfloor \right\},$$

$$S := \left\{ U_p \left(\frac{p^{\left\lfloor \frac{i}{p+1} \right\rfloor} \cdot b}{E_{p-1}^i} \right) \mod (q^{h'}, p^{m'}) : b \in B_i, i = 1, ..., \left\lfloor \frac{(p+1)m}{p-1} \right\rfloor \right\},$$

for an appropriate choice of h' and m' depending on m. We call Kb the *Katz basis*. Let d be its size and write $\mathrm{Kb} = \{v_1, ..., v_d\}$. Any overconvergent modular form of growth condition $\frac{1}{p+1}$, when reduced modulo $(q^{h'p}, p^{m'})$, can be expressed as a linear combination in Kb.

Let E and T be the $d \times h'$ matrices formed by taking the elements of Kb and S respectively and looking at the first h' terms in their q-expansions. Compute the $d \times d$ matrix A' such that $T = A'E$. Then, $A := A' \mod p^m$ is the representation of the operator U_p in the Katz basis. We write $A = [U_p]_{\mathrm{Kb}}$.

The advantage of this approach is that we only need to compute U_p once on the Katz basis and then we will be able to apply the Atkin operator as many times as we wish without having to actually use its original definition. Given an overconvergent modular form f of growth condition $\frac{p}{p+1}$, we can express it as a sum

$$f = \sum_i \alpha_i v_i \mod (q^{h'p}, p^{m'}). \tag{10}$$

Write $[f]_{\mathrm{Kb}} := (\alpha_1, ..., \alpha_d)$ and compute $A[f]_{\mathrm{Kb}}$. Letting γ_i denote the entries of $[U_p(f)]_{\mathrm{Kb}}$, we finally obtain

$$U_p(f) = \sum_i \gamma_i v_i \mod (q^{h'}, p^m). \tag{11}$$

Thus, we have $[U_p(f)]_{\mathrm{Kb}} = A[f]_{\mathrm{Kb}}$. For more details on the correctness Equation (11), see Section 2.2.2 of [Lau14] and the last paragraph of Section 3.2.1 in [Lau11].

Remark 3. Note that we let the overconvergent modular form f in Equation (10) have growth rate $\frac{p}{p+1}$ instead of just $\frac{1}{p+1}$. Although we can write an $\frac{1}{p+1}$-overconvergent modular form ϕ in the Katz basis, and $A = [U_p]_{\mathrm{Kb}}$ in the same basis, we cannot directly apply A to $[\phi]_{\mathrm{Kb}}$, as explained in [Lau14]. Indeed, the coefficients in the expansion of $[\phi]_{\mathrm{Kb}}$ will not decay fast enough (p-adically) for our calculations to be accurate and for Equation (11) to hold. This issue is entirely avoided when ϕ is $\frac{p}{p+1}$-overconvergent. Thus, when dealing with a $\frac{1}{p+1}$-overconvergent form ϕ, we have to compute $U_p(\phi)$ directly (without using the matrix representation A of U_p) to obtain a $\frac{p}{p+1}$-overconvergent form, thus improving its overconvergence and decay properties. After that, we may apply A to $[U_p(\phi)]_{\mathrm{Kb}}$.

To compute ordinary projections $e_{\mathrm{ord}} := \lim U_{p^{n!}}$ of overconvergent modular forms, we pick a big enough $R \in \mathbb{N}$ (cf. Algorithm 2.1 in [Lau14]) such that A^R represents e_{ord} to our desired level of precision. Given an overconvergent modular form f of growth condition $\frac{p}{p+1}$, written as $\sum_i \alpha_i v_i$ modulo $(q^{h'}, p^n)$, we compute $\gamma := A^R[f]_{\mathrm{Kb}}$ and let γ_i denote the entries of γ. Finally, we obtain

$$e_{\mathrm{ord}}(f) = \sum_i \gamma_i v_i \mod (q^{h'}, p^n).$$

3. Algorithmic methods

3.1. **Ordinary projections of nearly overconvergent modular forms.** For simplicity, let d denote the Serre operator $q\frac{\mathrm{d}}{\mathrm{d}q}$. Let g, h be two classical modular forms of weights ℓ, m respectively, and let $H := \mathrm{d}^{-(1+t)}(g^{[p]}) \times h$, for some integer t with $0 \leq t \leq \min\{\ell, m\} - 2$. We wish to compute

$$\mathcal{X} := e_{\mathrm{ord}}(H) = e_{\mathrm{ord}}\left(\mathrm{d}^{-(1+t)}(g^{[p]}) \times h\right).$$

The modular form $\mathrm{d}^{-(1+t)}(g^{[p]})$ has weight $\ell - 2(1+t)$, hence $\mathcal{X}$ has weight $\ell + m - 2t - 2$. The condition $0 \leq t \leq \min\{\ell, m\} - 2$ ensures that $\mathcal{X}, g$ and h are balanced, i.e. the largest weight is strictly smaller than the sum of the other two. If we had that $t = \ell - 2$, the form $H := \mathrm{d}^{-(1+t)}(g^{[p]}) \times h$ would have been overconvergent. This is because Theorem 2.2 still applies for negative powers of d, after depleting the modular form to avoid dividing by p, i.e. we have a map

$$\begin{aligned} M^{\mathrm{oc}}_{1+a}(B, \Gamma_1(N)) &\longrightarrow M^{\mathrm{oc}}_{1-a}(B, \Gamma_1(N)) \\ g &\mapsto \mathrm{d}^{-a} g^{[p]} = \sum_{p \nmid n} \frac{a_n(g)}{n^a} q^n, \end{aligned} \tag{12}$$

for all $a \geq 1$. So, in our case, H is not necessarily overconvergent and we cannot directly use the methods introduced in [Lau14] to compute the ordinary projection $e_{\mathrm{ord}}(H)$. However, H

is nearly overconvergent (Proposition 2.9 in [DR14]) and Theorem 2.5 tells us that

$$e_{\text{ord}}(H) = e_{\text{ord}}\left(\pi_{\text{oc}}(H)\right) = e_{\text{ord}}\left(\pi_{\text{oc}}\left(\mathrm{d}^{-(1+t)}(g^{[p]}) \times h\right)\right),$$

where π_{oc} is the overconvergent projection operator. Since $\pi_{\text{oc}}(H)$ is overconvergent, by definition, we can follow the methods described in [Lau14] to compute its ordinary projection, thus obtaining $e_{\text{ord}}\left(\pi_{\text{oc}}(H)\right) = e_{\text{ord}}(H)$. We therefore turn our attention to computing $\pi_{\text{oc}}(H)$. Note that we are not actually interested in taking the overconvergent projection of any nearly overconvergent modular form; we are specifically computing $\pi_{\text{oc}}(\mathrm{d}^{-(1+m)}(g^{[p]}) \times h)$. We therefore use a trick (see Theorem 3.1), that specifically applies to our setting.

Set $G := \mathrm{d}^{1-\ell} g^{[p]}$, it is an overconvergent modular form of weight $2-\ell$, as in Equation (12). Let $n = \ell - 2 - t \geq 0$ so that $\mathrm{d}^{-1-t} g^{[p]} = \mathrm{d}^n G$ and $\pi_{\text{oc}}\left(\mathrm{d}^{-1-t}(g^{[p]}) \times h\right) = \pi_{\text{oc}}((\mathrm{d}^n G) \times h)$. Consider the Rankin-Cohen bracket

$$[G, h]_n = \sum_{a,b \geq 0, a+b=n} (-1)^b \binom{(2-\ell)+n-1}{b} \binom{m+n-1}{a} \mathrm{d}^a(G)\mathrm{d}^b(h). \tag{13}$$

Note that the individual terms in this sum are all p-adic modular forms of weight $\ell+m-2t-2$ that are not necessarily overconvergent. However, the entire sum $[G, h]_n$ is overconvergent. It turns out that the Rankin-Cohen bracket is closely related to the overconvergent projection operator.

Theorem 3.1. *Let ϕ_1, ϕ_2 be overconvergent modular forms of weights κ_1 and κ_2 respectively, then, for all $s \geq 0$,*

$$[\phi_1, \phi_2]_s = \binom{\kappa_1 + \kappa_2 + 2s - 2}{s} \pi_{oc}((d^s \phi_1) \times \phi_2).$$

This follows from Section 4.4 of [LSZ20] (see also Theorem 1 in [Lan08]). We thus obtain the following Corollary.

Corollary 3.2. *We can relate $[G, h]_n$ and $\pi_{oc}((d^n G) \times h)$ as follows*

$$[G, h]_n = \binom{-\ell + m + 2n}{n} \pi_{oc}((d^n G) \times h).$$

Thus, we can simply compute $[G, h]_n$ using Equation (13) to obtain $\pi_{\text{oc}}(H)$.

Remark 4. Note that we had to pass through G instead of using g directly as we cannot have the subscript s of the Rankin-Cohen bracket $[\cdot, \cdot]_s$ be negative. Moreover, since the modular forms $\mathcal{X}, g, h$ are balanced, $\binom{-\ell+m+2n}{n}$ cannot be zero.

3.2. **Eigenspace σ projections.** In the previous sections, we have seen how to compute ordinary projections, i.e. projections over the space of overconvergent modular forms of slope zero. We now consider taking more general projections

For all $\alpha \in \mathbb{Q} \cup \{\infty\}$, the U_p equivariant decomposition of $M_k^{\text{oc}}(N)$, described in Equation (8), allows us to express any form H as a sum $H = F_\alpha + F$, where $F_\alpha \in M_k^{\text{oc}}(N)^{\text{slope } \alpha}$ and $F \in X_\alpha$. We call F_α the projection of H onto the space of slope α, or the slope α projection of H. Consider now the eigenspace associated to a single eigenvalue σ such that $\text{val}_p(\sigma) = \alpha$. We will explain how to project modular forms onto such an eigenspace. This method has been used in [DL21] and is based on an insight of David Loeffler (see the last paragraph of

Section 6.3 of [LSZ20]). We call such a projection the *eigenspace σ projection*. This can be seen as a special case of the slope α projection, as these two notions would agree in the case where U_p only has one eigenvalue σ of valuation α.

Let A denote the matrix computed in Section 2.3, representing the U_p operator acting on the Katz basis of $M_k^{\mathrm{oc}}(\mathbb{Z}_p, N, \chi; \frac{1}{p+1})$. Let σ be an eigenvalue for U_p and let $M = M_\sigma := A - \sigma\mathrm{Id}$. Put M_σ in *Smith normal form*, i.e. let P and Q be invertible matrices such that

$$QM_\sigma P = D = \mathrm{diag}\left(a_1(\sigma), ..., a_{s-1}(\sigma), a_s(\sigma)\right),$$

where $a_1(\sigma)|a_2(\sigma)|...|a_s(\sigma)$. We now remark that $a_s(\sigma)$ should be zero, as σ is an eigenvalue for U_p. However, A is only an approximation for U_p. More precisely, $A \in M_{d\times d}(\mathbb{Z}/p^m\mathbb{Z})$ is equal to U_p modulo p^m. And so, $a_s(\sigma)$ will only be zero in $\mathbb{Z}/p^m\mathbb{Z}$. Moreover, the case $a_{s-1}(\sigma) = 0$ happens precisely when σ has multiplicity (as an eigenvalue of U_p) more than one. Assume henceforth that we are dealing with an eigenvalue σ of multiplicity one, i.e. that the σ-eigenspace is one-dimensional.

From now on, we will assume the spectral expansion formula given by Equation (9). The spectral expansion conjecture [GM95] is widely believed to be true and has been proven in the case where $p = 2$, $N = 1$ and $5/12 < r < 7/12$ (cf. [Loe07]). Our algorithm for eigenspace σ projections will thus work under the assumption that this conjecture holds.

Let f_σ be an eigenform lying in the one-dimensional σ-eigenspace. Let $\pi := \pi_\sigma$ denote the last row of $Q \in M_{d\times d}(\mathbb{Z}/p^m\mathbb{Z})$, i.e. $\pi_i = Q_{i,d}$ for $i = 1, ..., d$. We call π the *projector* to f_σ. The reason for this will become clear in the following.

Proposition 3.3. *The projector π_σ is orthogonal to all $p/(p+1)$-overconvergent modular forms (written in the Katz basis) not in the σ-eigenspace.*

Proof. As we are working with $p/(p+1)$-overconvergent modular forms, we will be able to represent the action of U_p on them by the matrix A given in Section 2.3. We start with the simplest case. Let f_s be an eigenform of U_p with eigenvalue s, such that $s \neq \sigma$. Then, $M[f_s]_{\mathrm{Kb}} = (A - \sigma\mathrm{Id})[f_s]_{\mathrm{Kb}} = (s - \sigma)[f_s]_{\mathrm{Kb}}$. Hence,

$$Q(s - \sigma)[f_s]_{\mathrm{Kb}} = QM[f_s]_{\mathrm{Kb}} = DP^{-1}[f_s]_{\mathrm{Kb}}. \tag{14}$$

Since π is the last row of Q and the last row of D is completely zero, Equation (14) gives

$$(s - \sigma)\pi[f_s]_{\mathrm{Kb}} = \pi(s - \sigma)[f_s]_{\mathrm{Kb}} = 0. \tag{15}$$

As $s \neq \sigma$, we must have $\pi[f_s]_{\mathrm{Kb}} = 0$, up to a certain level of precision, as is explained in Remark 5. This shows that any eigenform of U_p, with eigenvalue of different norm than the norm of σ, is orthogonal to π.

A similar argument applies to generalized eigenform. Let F_s be a generalized eigenform for the eigenvalue s, again with $s \neq \sigma$. There exists some minimal integer $r \in \mathbb{N}$ such that $(A - s\mathrm{Id})^r[F_s]_{\mathrm{Kb}} = 0$. Let $M_s := A - s\mathrm{Id}$, so that $M_s^r[F_s]_{\mathrm{Kb}} = 0$. Then,

$$\left(M - M_s\right)^r [F_s]_{\mathrm{Kb}} = M\sum_{i=0}^{r-1}\binom{r}{i}(-1)^i M^{r-1-i}M_s^i[F_s]_{\mathrm{Kb}}.$$

Therefore, $\left(M - M_s\right)^r [F_s]_{\mathrm{Kb}} = MC[F_s]_{\mathrm{Kb}}$, where $C := \sum_{i=0}^{r-1}\binom{r}{i}(-1)^i M^{r-1-i}M_s^i$. Now, $\left(M - M_s\right)^r = (s-\sigma)^r\mathrm{Id}$, hence

$$(s - \sigma)^r \cdot Q[F_s]_{\mathrm{Kb}} = Q\left(M - M_s\right)^r [F_s]_{\mathrm{Kb}} = QMC[F_s]_{\mathrm{Kb}} = DP^{-1}C[F_s]_{\mathrm{Kb}}. \tag{16}$$

And as above, Equation (16) gives

$$(s-\sigma)^r \pi [F_s]_{\mathrm{Kb}} = 0. \tag{17}$$

Finally, since $s \neq \sigma$, we have $\pi[F_s]_{\mathrm{Kb}} = 0$, up to a certain level of precision (see Remark 5). That is, π must be orthogonal to all overconvergent modular forms not in the σ-eigenspace. □

Remark 5. It is crucial in Equations (15) and (17) that we are working over $\mathbb{Z}_p$ in order to conclude that $\pi[f_s]_{\mathrm{Kb}}$ and $\pi[F_s]_{\mathrm{Kb}}$ are zero. However, in practice, we are working over $\mathbb{Z}/p^m\mathbb{Z}$ for some $m \in \mathbb{Z}$. So Equation (17) actually becomes $p^m | (s-\sigma)^r \pi [f_s]_{\mathrm{Kb}}$, which does not necessarily imply that $p^m | \pi [f_s]_{\mathrm{Kb}}$. Therefore, there is a loss of precision of $r \cdot \mathrm{val}_p(s-\sigma)$. This loss of precision can be bounded above by looking at the largest non-zero entry of D, since $\mathrm{val}_p(s-\sigma) \leq \max_i \mathrm{val}_p(D_{i,i})$. To see this, using Equation (14), write

$$(s-\sigma) \ \mathrm{row}_i(Q) \cdot [f_s]_{\mathrm{Kb}} = D_{i,i} \ \mathrm{row}_i(P^{-1}) \cdot [f_s]_{\mathrm{Kb}}.$$

We now explain how to compute the projection $e_{\text{eigenspace } \sigma}(H)$ of an overconvergent modular form H in $M_k^{\mathrm{oc}}(\mathbb{Z}_p, N, \chi; \frac{p}{p+1})$ over the σ-eigenspace. First, we know that

$$H = \rho f_\sigma + \sum_{s \neq \sigma} F_s, \tag{18}$$

for some constant ρ, since we are assuming that the σ-eigenspace is one dimensional. This gives us $\pi \cdot [H] = \rho(\pi \cdot [f_\sigma])$. This is why we call π the *projector* to f_σ. Now, since π is non trivial, it cannot be orthogonal to all modular forms, so $\pi \cdot [f_\sigma]$ cannot also be zero. We hence deduce the following formula for the projection of H over f_σ:

$$\lambda_{f_\sigma}(H) := \rho = \frac{\pi \cdot [H]_{\mathrm{Kb}}}{\pi \cdot [f_\sigma]_{\mathrm{Kb}}}. \tag{19}$$

More formally, the projection operator λ_{f_σ} over f_σ is the unique Hecke-equivariant linear functional that factors through the Hecke eigenspace associated to f_σ and is normalized to send f_σ to 1 (cf. Definition 2.7 of [Loe18]). This gives us an associated idempotent operator $e_{f_\sigma}(\cdot) := \lambda_{f_\sigma}(\cdot) \ f_\sigma$. Since we are assuming that the σ-eigenspace is one dimensional, we have $e_{\text{eigenspace } \sigma}(H) = e_{f_\sigma}(H)$.

As explained in Remark 3, this holds under the assumption that H has growth condition $\frac{p}{p+1}$. In the case where H has growth condition $\frac{1}{p+1}$, we need to first apply the Atkin operator to H to obtain a modular form $U_p(H)$ of growth rate $\frac{p}{p+1}$, as in Remark 3. Indeed, write H as a sum $H = \rho f_\sigma + \sum_{s \neq \sigma} F_s$, as in Equation (18). Then,

$$U_p(H) = \rho\sigma f_\sigma + \sum_{s \neq \sigma} U_p(F_s).$$

Since the action of U_p preserves the eigenspaces of $M_k^{\mathrm{oc}}(N)$, we get that $\pi \cdot [U_p(F_s)]_{\mathrm{Kb}} = 0$ for $s \neq \sigma$, so $\pi \cdot [U_p(H)]_{\mathrm{Kb}} = \rho\sigma \ \pi \cdot [f_\sigma]_{\mathrm{Kb}}$. Finally,

$$\lambda_{f_\sigma}(U_p(H)) = \rho\sigma = \sigma \ \lambda_{f_\sigma}(H).$$

We thus obtain

$$\lambda_{f_\sigma}(H) = \frac{\pi \cdot [U_p(H)]_{\mathrm{Kb}}}{\sigma \ \pi \cdot [f_\sigma]_{\mathrm{Kb}}}. \tag{20}$$

Remark 6. In the case where the eigenvalue σ has multiplicity r greater than one, the eigenspace associated to σ will contain eigenforms other than the one we are projecting on. The method we are presenting here will thus not work because the projector $e_{\text{eigenspace } \sigma}$ over σ-eigenspace is not equal to e_{f_σ} anymore. In this case, one needs to use the last r rows of Q and the other Hecke operators in order to find a system of equations to solve and obtain $\lambda_{f_\sigma}(H)$.

As a simple example, assume that we already have a basis for the σ-eigenspace consisting of normalized Hecke eigenforms $\{\mathfrak{f}_1, ..., \mathfrak{f}_r\}$, with $\mathfrak{f}_1 = f_\sigma$. We then express the eigenspace σ projection of H as a linear combination $\sum_j a_j \mathfrak{f}_j$. Using the last r rows $\pi_1, ..., \pi_r$ of Q, we obtain a system of equations $\pi_i \cdot [H] = \sum_j a_j \ \pi_i \cdot [\mathfrak{f}_j]$. This can easily be solved in order to find $a_1 = \lambda_{f_\sigma}(H)$. The author has not yet implemented this method.

4. A symmetric p-adic symbol for triples of modular forms

To a modular form ϕ of weight two and level N, one can associate a differential $\omega_\phi \in H^1_{\mathrm{dR}}(X_1(N))$. In general, given a modular form ϕ of weight $r+2$ and level $\Gamma_1(N)$, one can associate to it a differential $\omega_\phi \in \mathrm{Fil}^{r+1} H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p)$, where $\mathcal{E}$ is the universal generalised elliptic curve fibered over $X_1(N)$, and $\mathcal{E}^r$ is the Kuga-Sato variety as in [Sch90]. Note that the ϕ-isotypic component of $H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p)$, denoted $H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p)_\phi$ is two dimensional. Assume now that ϕ is ordinary at p. This implies the existence of a one dimensional subspace (the *unit root subspace*) on which the Frobenius endomorphism acts as multiplication by a p-adic unit. We can then pick a unique element $\eta^{\text{u-r}}_\phi$ in this unit root subspace to extend $\{\omega_\phi\}$ to a basis $\{\omega_\phi, \eta^{\text{u-r}}_\phi\}$ such that $\langle \omega_\phi, \eta^{\text{u-r}}_\phi \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the alternating Poincaré duality pairing on $H^{r+1}_{\mathrm{dR}}(\mathcal{E}^r/\mathbb{C}_p)$.

Let f, g, h be three cuspidal eigenforms of level N, respective weights k, ℓ, m and respective characters χ_f, χ_g, χ_h. Fix a prime $p \geq 5$ and assume that $p \nmid N$, that $\chi_f \chi_g \chi_h = 1$ and that the weights k, ℓ, m are balanced, i.e. the largest one is strictly smaller than the sum of the other two. The assumption $p \geq 5$ is purely for simplicity and could potentially be relaxed at the cost of some extra care. Let $\alpha_{f,p}, \beta_{f,p}$ be the roots of the *Hecke polynomial* $x^2 - a_p(f)x + p^{k-1}\chi_f(p)$. Assume that the modular forms f, g and h are *ordinary* and *regular* at p, so that $\alpha_{f,p}, \alpha_{g,p}$ and $\alpha_{h,p}$ are units. Let f_α and f_β be the p-stabilizations of f given by Equation (1). Let

$$t := \frac{\ell + m - k - 2}{2} \geq 0, \qquad c := \frac{k + \ell + m - 2}{2}.$$

We may then define the Euler factors:

$$\begin{aligned} \mathcal{E}(f,g,h) &:= (1 - \beta_f \alpha_g \alpha_h p^{-c})(1 - \beta_f \alpha_g \beta_h p^{-c})(1 - \beta_f \beta_g \alpha_h p^{-c})(1 - \beta_f \beta_g \beta_h p^{-c}); \\ \tilde{\mathcal{E}}(f,g,h) &:= (1 - \alpha_f \alpha_g \alpha_h p^{-c})(1 - \alpha_f \alpha_g \beta_h p^{-c})(1 - \alpha_f \beta_g \alpha_h p^{-c})(1 - \alpha_f \beta_g \beta_h p^{-c}); \\ &\mathcal{E}_0(f) := 1 - \beta_f^2 \chi_f^{-1}(p) p^{1-k}; \qquad \tilde{\mathcal{E}}_0(f) := 1 - \alpha_f^2 \chi_f^{-1}(p) p^{1-k}; \\ &\mathcal{E}_1(f) := 1 - \beta_f^2 \chi_f^{-1}(p) p^{-k}; \qquad \tilde{\mathcal{E}}_1(f) := 1 - \alpha_f^2 \chi_f^{-1}(p) p^{-k}. \end{aligned} \tag{21}$$

4.1. **The Garrett-Rankin triple product p-adic L-function.** Following Section 2.6 of [DR14], let $\Gamma := 1 + pN\mathbb{Z}_p$ and let $\Lambda := \mathcal{O}[[\Gamma]]$ be the completed group ring of Γ. Let also $\Lambda' := \mathrm{Frac}(\Lambda)$. Lastly, let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be Hida families, with coefficients in finite flat extensions $\Lambda_f, \Lambda_g, \Lambda_h$ of Λ, interpolating f, g and h at the weights k, ℓ and m. The existence of such families is guaranteed by Hida's construction in [Hid86]. Let $\mathbf{f}^* := \mathbf{f} \otimes \chi_f^{-1}$, and note that for classical points x (in $\mathbb{Z}$) we have $(f^*)_x = (f_x)^*$. We write $\kappa(x)$ for the weight of f_x.

Assume now that the action of the Hecke algebra on the ordinary subspace in weight k is semi-simple (Assumption $(S3)$ on p. 222 in [Hid93]). This is the case for N square-free, since $k \geq 2$, as is described in [Lau14]. This allows us to define the following operator. Given an ordinary eigenform F and an ordinary overconvergent modular form G we let $c(F, G)$ denote the coefficient of F appearing in the expression of G as a linear combination of ordinary (normalized) eigenforms.

Definition 4.1 (Lemma 2.19, [DR14])**.** The Garrett-Rankin triple product p-adic L-function attached to the triple $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of Λ-adic modular forms is the unique $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ in $\Lambda'_f \otimes_\Lambda (\Lambda_g \otimes \Lambda_h \otimes \Lambda)$ such that at classical balanced points (x, y, z) we have

$$\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z) := c\left(f_x^{*(p)}, e_{\mathrm{ord}}(\mathrm{d}^{-1-t}({g_y}^{[p]}) \times h_z)\right), \tag{22}$$

where $t := \frac{\kappa(y)+\kappa(z)-\kappa(x)-2}{2}$, $f_x^* := f_x \otimes \chi_f^{-1}$ is the dual of f_x and $f_x^{*(p)}$ is the ordinary p-stabilization of f_x^*. We write $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h}) := c\left(\mathbf{f}^*, e_{\mathrm{ord}}(\mathrm{d}^\bullet \mathbf{g}^{[p]} \times \mathbf{h})\right)$ for notational brevity.

Remark 7. We project over $\mathbf{f}^* := \mathbf{f} \otimes \chi_f^{-1}$ instead of $\mathbf{f}$ in Definition 4.1 because $e_{\mathrm{ord}}(\mathrm{d}^\bullet \mathbf{g}^{[p]} \times \mathbf{h})$ has character $\chi_g \chi_h = \chi_f^{-1}$.

We can express the Garrett-Rankin triple product p-adic L-function $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at classical balanced points as $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z) := \lambda_{f^*_{x,\alpha}}(d^{-1-t} g_y^{[p]} \times h_z)$, where $f^*_{x,\alpha} := (f^*_x)_\alpha$ is the ordinary p-stabilization of the dual of f_x and λ is the projection operator from Section 3.2.

Equation (22) reveals that in order to experimentally compute values for $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z)$ the main ingredient is the computation of ordinary projections of p-adic modular forms. In [Lau14], parts of which have been summarized here in Section 2.3, the author explains how to calculate the ordinary projections of overconvergent modular forms, and is thus able to compute special values of the Garrett-Rankin triple product p-adic L-function, for balanced weights (k, ℓ, m) satisfying $k = 2 + m - \ell$. Indeed, this condition guarantees that $\mathrm{d}^{-1-t}({g_\ell}^{[p]}) \times h_m$ will be overconergent, thus the code and the theory in [Lau14] are enough.

In general, however, when the weights (k, ℓ, m) are only balanced, $\mathrm{d}^{-1-t}({g_y}^{[p]}) \times h_z$ is only *nearly overconvergent*. We therefore need to use the generalizations we introduced in Section 3.1 in order to compute ordinary projections of nearly overconvergent modular forms, thus being able to compute the Garrett-Rankin triple product p-adic L-function for any balanced classical weights.

In Section 3 of [DR14], the authors construct the generalized Gross-Kudla-Schoen diagonal cycle $\Delta := \Delta_{k,\ell,m}$ for a triple of balanced classical weights (k, ℓ, m). More precisely, this cycle is an element of the Chow group $\mathrm{CH}^{r+2}(W)_0$ where $W := \mathcal{E}^{k-2} \times \mathcal{E}^{\ell-2} \times \mathcal{E}^{m-2}$ and $r := (k + \ell + m)/2 - 3$. One can check from Definition 3.3 of [DR14] that $\Delta_{k,\ell,m}$ indeed has codimension $r + 2$. Let

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(W)_0 \longrightarrow \mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W)^\vee$$

be the p-adic Abel-Jacobi map (cf. Section (1.2) of [Nek00] or [Bes00]). Darmon and Rotger then show, in Theorem 3.14 of [DR14], that

$$\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h) = (-1)^{t+1} t! \frac{\mathcal{E}_1(f)}{\mathcal{E}(f, g, h)} \langle \eta_f^{\text{u-r}}, d^{-1-t} g^{[p]} \times h \rangle, \tag{23}$$

where $t := \frac{\ell+m-k-2}{2}$. In Equation (23), $\omega_g \in H^{\ell-1}_{\mathrm{dR}}(\mathcal{E}^{\ell-2}/\mathbb{C}_p)_g$ and $\omega_h \in H^{m-1}_{\mathrm{dR}}(\mathcal{E}^{m-2}/\mathbb{C}_p)_h$ denote the differentials associated to the forms g and h that we introduced at the start of Section 4. Furthermore, $\eta_f^{\text{u-r}}$ denotes the element lying in the unit root space of $H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{k-2}/\mathbb{C}_p)_{f^*}$ such that $\langle \omega_f, \eta_f^{\text{u-r}}\rangle = 1$, where $\omega_f \in H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{k-2}/\mathbb{C}_p)_{f^*}$ is the differential associated to f^*. Note that although our notation for ω_g and ω_h is the same as the one introduced at the start of Section 4, our notation for ω_f and $\eta_f^{\text{u-r}}$ is not. Indeed, the roles of f and f^* have been switched in ω_f and $\eta_f^{\text{u-r}}$; but g and h are still the same and have not been replaced by their duals in ω_g and ω_h. This choice is necessary, as explained in Remark 7, and is consistent with the notation used in [DR14]. Finally, given the cohomology classes $\eta_f^{\text{u-r}} \in H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{k-2})$, $\omega_g \in H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{\ell-2})$ and $\omega_h \in H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{m-2})$, we can view the product $\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h$ in Equation (23) as an element of $\mathrm{H}^{2r+3}_{\mathrm{dR}}(W)$ thanks to the Künneth decomposition.

We now, as in Theorem 5.1 of [DR14], provide an alternative way to express the Garrett-Rankin triple product p-adic L-function by relating it to the generalized Gross-Kudla-Schoen diagonal cycle as follows.

Proposition 4.2. *We have*

$$\mathrm{AJ}_p(\Delta)(\eta_f^{u\text{-}r} \otimes \omega_g \otimes \omega_h) = (-1)^t t! \frac{\mathcal{E}_0(f)\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \lambda_{f^*_\alpha}(d^{-1-t} g^{[p]} \times h).$$

Proof. By Theorem 3.14 in [DR14], we have

$$\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h) = \left\langle \eta_f^{\text{u-r}}, -\frac{(-1)^t t! \mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} e_{f^*,\mathrm{ord}}(d^{-1-t} g^{[p]} \times h) \right\rangle.$$

Note that we write $\langle \eta_f^{\text{u-r}}, \phi\rangle$ here to mean $\langle \eta_f^{\text{u-r}}, \omega_\phi\rangle$ by abuse of notation. We observe that the f^*-isotypic component of $e_{\mathrm{ord}}(d^{-1-t} g^{[p]} \times h)$ is $\lambda_{f^*_\alpha}(d^{-1-t} g^{[p]} \times h) f^*_\alpha$, because we can express $e_{\mathrm{ord}}(d^{-1} g^{[p]} \times h)$ as

$$\lambda_{f^*_\alpha}(d^{-1-t} g^{[p]} \times h) f^*_\alpha + (\text{terms attached to other ordinary forms}).$$

Therefore,

$$\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h) = (-1)^{t+1} t! \frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \lambda_{f^*_\alpha}(d^{-1-t} g^{[p]} \times h) \left\langle \eta_f^{\text{u-r}}, f^*_\alpha \right\rangle.$$

Next, $f^*_\alpha = \mathcal{E}_0(f) e_{\mathrm{ord}}(f^*)$ by applying the proof of Lemma 4.1 to f^* instead of f, thus by Proposition 2.11 in [DR14],

$$\left\langle \eta_f^{\text{u-r}}, f^*_\alpha \right\rangle = \mathcal{E}_0(f) \left\langle \eta_f^{\text{u-r}}, e_{\mathrm{ord}}(f^*) \right\rangle = \mathcal{E}_0(f) \left\langle \eta_f^{\text{u-r}}, f^* \right\rangle = -\mathcal{E}_0(f),$$

as $\left\langle \eta_f^{\text{u-r}}, f^* \right\rangle = -\left\langle f^*, \eta_f^{\text{u-r}} \right\rangle = -1$ by definition of $\eta_f^{\text{u-r}}$. We finally obtain

$$\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h) = (-1)^t t! \frac{\mathcal{E}_0(f)\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \lambda_{f^*_\alpha}(d^{-1-t} g^{[p]} \times h).$$

□

Corollary 4.3. *The Garrett-Rankin triple product p-adic L-function can be written, at classical balanced points (k,ℓ,m), as*

$$\mathscr{L}_p(\mathbf{f},\mathbf{g},\mathbf{h})(k,\ell,m) = \frac{(-1)^t}{t!} \frac{\mathcal{E}(f,g,h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \mathrm{AJ}_p(\Delta)(\eta_f^{u\text{-}r} \otimes \omega_g \otimes \omega_h). \tag{24}$$

Equation (22), provides us with a compact way to express the Garrett-Rankin p-adic L-function. Equation (24) on the other hand connects it to the Abel Jacobi map and provides us with the right insight in order to define a new natural symbol, based on the Garrett-Rankin triple product p-adic L-function, which we expect to have nice symmetry properties.

4.2. **A new p-adic triple symbol $(f, g, h)_p$.** We continue working in the same setup as the previous section. The differentials $\omega_g \in H^{\ell-1}_{\mathrm{dR}}(\mathcal{E}^{\ell-2}/\mathbb{C}_p)_g$ and $\omega_h \in H^{m-1}_{\mathrm{dR}}(\mathcal{E}^{m-2}/\mathbb{C}_p)_h$ are the basis elements that we introduced at the start of Section 4. Similarly to Section 4.1, ω_f will denote the differential associated to f^*, and not f. This choice is necessary, as explained in the second part of Remark 7.

Our goal is to define define a new quantity involving $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ instead of $\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h)$, and believe that this alternative should have nice symmetry properties. We investigate such properties further in the following sections. Before defining our new symbol, we first provide a way to express $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ in terms of projections onto isotypic spaces, similarly to Proposition 4.2. Let

$$\ell_{fgh,\alpha} := \lambda_{f^*_\alpha}\left(\mathrm{d}^{-1-t}(g^{[p]}) \times h\right); \qquad \ell_{fgh,\beta} := \lambda_{f^*_\beta}\left(\pi_{\mathrm{oc}}\left(\mathrm{d}^{-1-t}(g^{[p]}) \times h\right)\right). \tag{25}$$

Note that including π_{oc} before $\lambda_{f^*_\alpha}$ in (25) would be redundant, by Theorem 2.5.

Lemma 4.1. *Let f be a classical eigenform of weight k that is ordinary at p with $\mathrm{val}_p(\alpha_{f,p}) = 0$. Then, we have $e_{ord}(f) = \frac{1}{\mathcal{E}_0(f)} f_\alpha$ and $e_{slope\ k-1}(f) = \frac{1}{\tilde{\mathcal{E}}_0(f)} f_\beta$.*

Proof. We have by definition $f_\alpha(q) := f(q) - \beta f(q^p)$ and $f_\beta(q) := f(q) - \alpha f(q^p)$. So, $\alpha f_\alpha - \alpha f = \beta f_\beta - \beta f$. Hence, $\alpha f_\alpha - \beta f_\beta = (\alpha - \beta) f$. Thus, using the notation from (21), we have

$$e_{\mathrm{ord}}(f) = \frac{\alpha f_\alpha}{\alpha - \beta} = \frac{1}{\mathcal{E}_0(f)} f_\alpha, \qquad e_{\mathrm{slope}\ k-1}(f) = \frac{\beta f_\beta}{\beta - \alpha} = \frac{1}{\tilde{\mathcal{E}}_0(f)} f_\beta.$$

□

Theorem 4.4. *Let $t := \frac{\ell+m-k-2}{2}$. We may rewrite $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ as*

$$\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h) = (-1)^t t! \frac{\langle \omega_f, \phi(\omega_f)\rangle}{p^{k-1}} \left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)}\ \beta_{f^*}\ \ell_{fgh,\alpha} + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)}\ \alpha_{f^*}\ \ell_{fgh,\beta} \right).$$

Proof. Note that f^* is orthogonal to the kernel of e_{f^*}, so $\langle f^*, \phi\rangle = \langle f^*, e_{f^*}(\phi)\rangle$ only depends on the projection $e_{f^*}(\phi)$ of ϕ, for any modular form ϕ. Adapting this to our notation, we obtain $\langle \omega_f, \phi\rangle = \langle \omega_f, e_{f^*}(\phi)\rangle$, as ω_f, here, is the differential attached to f^*. Furthermore, $e_{f^*}(\phi)$ only depends on the overconvergent projection of ϕ. Indeed, $\phi - \pi_{\mathrm{oc}}(\phi)$ is purely nearly overconvergent (i.e. it has no overconvergent part) and will not lie in the f^*-isotypic space, as f^* is overconvergent. Lemma 4.1 tells us that f has only two slope components: an ordinary one and one of slope $k-1$. Namely, $f = \frac{1}{\mathcal{E}_0(f)} f_\alpha + \frac{1}{\tilde{\mathcal{E}}_0(f)} f_\beta$, and thus to project over the f^*-isotypic space, one needs to project over the components f^*_α and f^*_β. Adapting the proof of Proposition 4.2 for the case of $\mathrm{AJ}(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$, and using the notation $\xi(\omega_g, \omega_h)$ from [DR14] (see Equation (72) on p. 30), we write

$$\begin{aligned}
\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h) &= \langle \omega_f, \xi(\omega_g, \omega_h)\rangle \\
&= \langle \omega_f, e_{f^*,\mathrm{ord}}(\xi(\omega_g, \omega_h)) + e_{f^*,\mathrm{slope}\ k-1}(\xi(\omega_g, \omega_h))\rangle \\
&= \left\langle \omega_f, \frac{(-1)^t t! \mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} e_{f^*,\mathrm{ord}}(d^{-1-t} g^{[p]} \times h) \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \left\langle \omega_f, \frac{(-1)^t t! \tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} e_{f^*,\text{slope } k-1}\left(\pi_{\mathrm{oc}}(d^{-1-t} g^{[p]} \times h)\right) \right\rangle \\
&= (-1)^t t! \frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \lambda_{f^*_\alpha}(d^{-1-t} g^{[p]} \times h) \left\langle \omega_f, f^*_\alpha \right\rangle \\
&+ (-1)^t t! \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \lambda_{f^*_\beta}\left(\pi_{\mathrm{oc}}(d^{-1-t} g^{[p]} \times h)\right) \left\langle \omega_f, f^*_\beta \right\rangle.
\end{aligned}
$$

As $\langle \omega_f, f^* \rangle = \langle \omega_f, \omega_f \rangle = 0$, we can write

$$
\langle \omega_f, f^*_\alpha \rangle = \langle \omega_f, f^* - \beta_{f^*} V f^* \rangle = -\beta_{f^*} \langle \omega_f, \omega_{V f^*} \rangle = -\frac{\beta_{f^*}}{p^{k-1}} \langle \omega_f, \phi(\omega_f) \rangle.
$$

Similarly, $\left\langle \omega_f, f^*_\beta \right\rangle = -\frac{\alpha_{f^*}}{p^{k-1}} \langle \omega_f, \phi(\omega_f) \rangle$. This gives the desired result. □

We are now ready to write down our new candidate for a symmetric p-adic triple symbol.

Definition 4.5. Let f, g and h be three cuspidal modular forms of level N and respective weights k, ℓ and m which are ordinary at p. We define the p-adic triple symbol $(f,g,h)_p$ by

$$
(f,g,h)_p := (-1)^t t! \frac{\langle \omega_f, \phi(\omega_f) \rangle}{p^{k-1}} \left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \ \beta_{f^*} \ \ell_{fgh,\alpha} + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \ \alpha_{f^*} \ \ell_{fgh,\beta} \right). \tag{26}
$$

In Definition 4.5, we do not actually need g and h to be cuspidal nor ordinary at p, as Equation (26) is still defined when only f is. However, as we are interested in permuting the order of f, g and h, we often require them all to be cuspidal and ordinary at p. Thanks to Theorem 4.4, we may reformulate $(f,g,h)_p$ as follows.

Corollary 4.6. *We have* $(f,g,h)_p = \mathrm{AJ}_p(\Delta_{k,\ell,m})(\omega_f \otimes \omega_g \otimes \omega_h)$.

The right hand side of the equation in Corollary 4.6 appears to be symmetric in the variables f, g, h, and thus suggests that $(f,g,h)_p$ is symmetric.

4.3. **Symmetry properties of $(f,g,h)_p$.** We are interested in the behaviour of the p-adic triple symbol $(f,g,h)_p$ as we vary the order of f, g and h. Our experiments have shown that in some cases $(f,g,h)_p = (f,h,g)_p$; while in some other cases $(f,g,h)_p = -(f,h,g)_p$. This led us to the following result.

Theorem 4.7. *Let f, g, h be three cuspidal newforms of weights k, ℓ, m. Let $t := \frac{\ell+m-k-2}{2}$. We have the following relations:*

$$
\begin{aligned}
\mathscr{L}_p(f,g,h) &= (-1)^{t+1} \mathscr{L}_p(f,h,g), \\
(f,g,h)_p &= (-1)^{t+1} (f,h,g)_p,
\end{aligned}
$$

i.e. the parity of t determines the symmetry or anti-symmetry of $(f,*,*)_p$ and $\mathscr{L}_p(f,\cdot,\cdot)$.

Theorem 4.7 can be proven via an explicit calculation, expanding the definitions of the involved quantities in terms of Poincaré pairings and noticing that certain forms are exact and are thus in the kernel of the ordinary projection operator and must also vanish in cohomology. The full proof can be found in Section 4.3.1 of [Gha23]. We will not present it here, as an alternative proof for Theorem 4.7 can be obtained, based on that of Theorem 4.8, which we prove in a more general fashion, using the p-adic Abel Jacobi map.

We finally present our main theorem, proving the symmetry property of $(*,*,*)_p$, when permuting its inputs.

Theorem 4.8. *Let f, g, h be three cuspidal newforms of weights k, ℓ, m. Then $(f, g, h)_p$ satisfies the cyclic symmetry relation*

$$(f, g, h)_p = (-1)^k (g, h, f)_p = (-1)^m (h, f, g)_p.$$

In particular, when the weights are all even, $(f, g, h)_p$ is symmetric when its inputs are cyclically permuted.

Proof. Assume for simplicity that $\chi_f = \chi_g = \chi_h = 1$. We start with the case of weights $k = \ell = m = 2$. In this case, the diagonal cycle $\Delta_{2,2,2}$ is symmetric, as can easily be seen from Definition 3.1 in [DR14]. Recall that $\omega_f \otimes \omega_g \otimes \omega_h$ is given by the Künneth decomposition and is therefore composed of cup products. So by the properties of cup products, we have $\omega_f \otimes \omega_g = -\omega_g \otimes \omega_f$ and $\omega_f \otimes \omega_h = -\omega_h \otimes \omega_f$. We can thus write $\mathrm{AJ}_p(\Delta_{2,2,2})(\omega_f \otimes \omega_g \otimes \omega_h) = \mathrm{AJ}_p(\Delta_{2,2,2})(\omega_g \otimes \omega_h \otimes \omega_f)$.

For general weights k, ℓ, m, a variation of the above holds. We will first study the action of permuting the first two coordinates of $(f, g, h)_p$, then the action of permuting the second and third coordinates and finally combine them to obtain the desired result. We make our argument explicit using the functoriality properties of the p-adic Abel Jacobi map. Let $r_1 := k - 2, r_2 := \ell - 2, r_3 := m - 2, r := (r_1 + r_2 + r_3)/2$ and let s be the map going from $W := \mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}$ to $W' := \mathcal{E}^{r_2} \times \mathcal{E}^{r_1} \times \mathcal{E}^{r_3}$ that permutes the first and second terms. Then s induces permutations on the corresponding Chow groups and De Rham cohomology groups: we have a pushforward s_* on $\mathrm{CH}^{r+2}(W)_0$ and a dual pullback $s^{*,\vee}$ on $\mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W)^\vee$. The functoriality properties of the p-adic Abel Jacobi map with respect to correspondences (see Propositions 1, 2 & 4 (iii) in [EZZ82]) give us the commuting diagram

$$\begin{array}{ccc} \mathrm{CH}^{r+2}(W)_0 & \xrightarrow{\mathrm{AJ}_p} & \mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W)^\vee \\ \downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle s^{*,\vee}} \\ \mathrm{CH}^{r+2}(W')_0 & \xrightarrow{\mathrm{AJ}_p} & \mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W')^\vee. \end{array}$$

Thus, $\mathrm{AJ}_p s_* = s^{*,\vee}\mathrm{AJ}_p$. Given $Z \in \mathrm{CH}^{r+2}(W)_0$ and some $\omega \in \mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W')$, we get $\mathrm{AJ}_p(s_* Z)(\omega) = (s^{*,\vee}\mathrm{AJ}_p(Z))(\omega) = \mathrm{AJ}_p(Z)(s^*\omega)$. We can now apply this to the generalized Gross-Kudla-Schoen diagonal cycle $\Delta_{k,\ell,m}$ and take $\omega := \omega_g \otimes \omega_f \otimes \omega_h$. We see that the action of s^* on ω is given by $s^*(\omega_g \otimes \omega_f \otimes \omega_h) = (-1)^{(k-1)(\ell-1)}(\omega_f \otimes \omega_g \otimes \omega_h)$, by the skew symmetry of cup products (which are part of the Künneth decomposition). Furthermore, the action of s_* on $\Delta_{k,\ell,m}$ is given by $s_*\Delta_{k,\ell,m} = (-1)^{r+(r_1 r_2)}\Delta_{\ell,k,m}$. The proof of this is purely combinatorial: one needs to expand Definition 3.3 of $\Delta_{k,\ell,m} \in \mathrm{CH}^{r+2}(W)_0$ in [DR14] and permute two subsets of $\{1, ..., r\}$ of size r_1 and r_2 and intersection of size $r - r_3$. Finally, $r + r_1 r_2 + (k-1)(\ell-1) = (k + \ell - m)/2 \mod 2$, therefore we obtain the symmetry formula

$$(f, g, h)_p = (-1)^{(k+\ell-m)/2}(g, f, h)_p.$$

Similarly, $(f, g, h)_p = (-1)^{(\ell+m-k)/2}(f, h, g)_p$. Combining these two symmetry formulas gives $(f, g, h)_p = (-1)^k (g, h, f)_p$. □

5. Examples

5.1. **Computing Poincaré pairings.** Using our algorithms from Section 3, we can compute $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$ appearing in Equation (26). The only remaining factor in this equation that is non-trivial to calculate is the period $\Omega_f := \langle \omega_f, \phi(\omega_f) \rangle$.

When f is a newform (with rational coefficients) of weight 2, we can use the following trick to calculate Ω_f, as was done in Section 4 of [DL21]. Let E be the elliptic curve associated to f. The differential $\omega_f = f\frac{\mathrm{d}q}{q}$ corresponds to the differential $\omega_E := \frac{\mathrm{d}x}{y}$ of the elliptic curve E. Computing the Poincaré pairing $\langle \omega_f, \phi(\omega_f)\rangle$ now amounts to calculating $\langle \omega_E, \mathrm{Frob}(\omega_E)\rangle$, up to including the modular degree m_E of E: $\langle \omega_f, \phi(\omega_f)\rangle = m_E \langle \omega_E, \mathrm{Frob}(\omega_E)\rangle$. Let M be the matrix representing the action of Frobenius, up to precision p^m, on $\omega_E = \frac{\mathrm{d}x}{y}$ and $\eta_E := x\frac{\mathrm{d}x}{y}$. Then, $\langle \omega_E, M\omega_E\rangle = \langle \omega_E, M_{11}\omega_E + M_{21}\eta_E\rangle = M_{21}$ so that the period Ω_f is simply given by

$$\Omega_f = m_E M_{21} \mod p^m,$$

and the matrix M can be efficiently computed via Kedlaya's algorithm (cf. [Ked01]).

In the case where f has weight k strictly greater than 2, we cannot use the above trick anymore. Instead, we can exploit the symmetry of $(*,*,*)_p$ and the algorithms mentioned so far in this paper. Indeed, in order to calculate the period Ω_f, we first appropriately chose two auxiliary forms f_0 and φ, such that Ω_{f_0} is known or computable (e.g. when f_0 has weight 2). Then, using the symmetry relation of Theorem 4.8, we obtain $(f, f_0, \varphi)_p = (-1)^k (f_0, \varphi, f)_p$. The right hand side containing Ω_{f_0} is entirely known, whereas the left hand side is entirely computable except for Ω_f. We can thus recover the value of Ω_f.

This method is explained is great detail in Section 6.2 of [Gha23]. It is however simpler to illustrate it by means of examples. See in particular Examples 3, 5 and 6 in the next section.

5.2. **Symmetry relations for even weights.** We dedicate this section to gathering experimental evidence verifying Theorem 4.8, thus providing examples that demonstrate the correctness of our algorithms described in Section 3.

We begin with a simple case where all modular forms have weight 2. This only involves overconvergent modular forms, and we can compute Ω_f via Kedlaya's algorithm.

Example 1. Consider the space of newforms $S_2^{\mathrm{new}}(\mathbb{Q}, 57)$ of weight 2 and level 57. Let f, g and h be the cuspidal newforms in $S_2^{\mathrm{new}}(\mathbb{Q}, 57)$ given by:

$$\begin{aligned} f &= q - 2q^2 - q^3 + 2q^4 - 3q^5 + 2q^6 - 5q^7 + q^9 + 6q^{10} + q^{11} + ..., \\ g &= q + q^2 + q^3 - q^4 - 2q^5 + q^6 - 3q^8 + q^9 - 2q^{10} + ..., \\ h &= q - 2q^2 + q^3 + 2q^4 + q^5 - 2q^6 + 3q^7 + q^9 - 2q^{10} - 3q^{11} + ... \end{aligned}$$

Fix $p := 5$ and let $f_{\alpha_{f,p}}$ and $f_{\beta_{f,p}}$ denote the p-stabilizations of f at p. Then f, g and h are regular and ordinary at p. Using the algorithms described in Section 3, we compute the quantities $\ell_{fgh,\alpha}, \ell_{fgh,\beta}, \ell_{ghf,\alpha}, \ell_{ghf,\beta}, \ell_{hfg,\alpha}, \ell_{hfg,\beta}$ and obtain

$$\begin{aligned} \ell_{fgh,\alpha} &= -37749288269657878165114377581799159847389728556133488701497403875138 06 \mod 5^{100}, \\ \ell_{fgh,\beta} &= -1600120463087968696799905890349018972704454279824366881678828640068804 \cdot 5^{-1} \mod 5^{99}, \\ \ell_{ghf,\alpha} &= 3414089135682117556340078214096537672013164967359802729338191598002457 \cdot 5 \mod 5^{101}, \\ \ell_{ghf,\beta} &= 3193246879655120717163186432727961266470176374874741691284821764797 03 \mod 5^{100}, \\ \ell_{hfg,\alpha} &= 3386642279338565749426053729955310360166771341172640348803607194424548 \cdot 5^{-1} \mod 5^{99}, \\ \ell_{hfg,\beta} &= -1362182692510584292629393424534010351729144263363030199124032659953338 \mod 5^{100}, \\ \ell_{fhg,\alpha} &= 3774928826965787816511437758179915984738972855613348870149740387513806 \mod 5^{100}, \\ \ell_{fhg,\beta} &= 8806793175264059302644094388111179312424900110049379013283342553031 79 \cdot 5^{-1} \mod 5^{99}, \end{aligned}$$

$$\ell_{gfh,\alpha} = -3414089135682117556340078214096537672013164967359802729338191598002457 \cdot 5 \bmod 5^{101},$$
$$\ell_{gfh,\beta} = 1316444872164870993756743549790237920953950571465279247158114744839078 \bmod 5^{100},$$
$$\ell_{hgf,\alpha} = -1808920468896542138602596599389737900820358470954594339263049333096423 \cdot 5^{-1} \bmod 5^{99},$$
$$\ell_{hgf,\beta} = -1848796736101022160118506527593042717532675210104737039492910699421662 \bmod 5^{100}.$$

Note that we indeed have $\ell_{fgh,\gamma} = -\ell_{fhg,\gamma}$. In order to experimentally verify the symmetry property of Theorem 4.8, we will now compute the periods $\Omega_f, \Omega_g, \Omega_h$. Using Kedlaya's algorithm, we obtain

$$\Omega_f = 2950568119913096262656125583897759935633329467905628286532407351406 8 \cdot 5^2 \bmod 5^{100},$$
$$\Omega_g = -1591334613811759017043393805285841683927462644737009846197261394355 77 \cdot 5 \bmod 5^{100},$$
$$\Omega_h = 7841489370896526206130486010581886879377965958702903183489820661963 9 \cdot 5^2 \bmod 5^{100}.$$

Finally, putting everything together we obtain

$$(f,g,h)_p = 5871767952506844465150908265973598858284513190743516082327198557652 \cdot 5^2 \bmod 5^{100},$$
$$(g,h,f)_p = 9422418933726016667126450757764565658168363392295409261659843879202 7 \cdot 5^2 \bmod 5^{100},$$
$$(h,f,g)_p = 3289891947310332799617949286058028385324298690113993388362334485576 52 \cdot 5^2 \bmod 5^{100}.$$

And we can check that all these values agree modulo 5^{97}.

The next example involves nearly overconvergent modular forms, thus utilizing the full power of Section 3.2. Since we cannot compute the period Ω_f as in Example 1, we use the following ratio trick. We introduce an extra modular form h_2 and check for

$$\frac{(f,g,h_1)_p}{(f,g,h_2)_p} \stackrel{?}{=} \frac{(g,h_1,f)_p}{(g,h_2,f)_p}. \tag{27}$$

This has the advantage of bypassing the calculation of the periods Ω_f and Ω_g, as they appear in both the numerator and the denominator of Equation (27).

Example 2. Let $f, g, h, h_2, h_3 \in S_4(\mathbb{Q}, 45)$ be the cuspidal newforms given by:

$$\begin{aligned} f &= q - q^2 - 7q^4 - 5q^5 - 24q^7 + 15q^8 + 5q^{10} - 52q^{11}..., \\ g &= q - 3q^2 + q^4 + 5q^5 + 20q^7 + 21q^8 - 15q^{10} + 24q^{11}..., \\ h &= q + 4q^2 + 8q^4 + 5q^5 + 6q^7 + 20q^{10} - 32q^{11} + ..., \\ h_2 &= q - 5q^2 + 17q^4 + 5q^5 - 30q^7 - 45q^8 - 25q^{10} - 50q^{11} + ..., \\ h_3 &= q + 5q^2 + 17q^4 - 5q^5 - 30q^7 + 45q^8 - 25q^{10} + 50q^{11} + ... \end{aligned}$$

For $p := 17$, we have $a_{17}(f) \cdot a_{17}(g) \cdot a_{17}(h) \cdot a_{17}(h_2) \cdot a_{17}(h_3) \neq 0$. When considering the p-adic symbols $(\phi_1, \phi_2, \phi_3)_p$, for $\phi_i \in \{f, g, h, h_2, h_3\}$ distinct and up to permutations, we have ten potential values to compute. Up to precision 30 (i.e. in $\mathbb{Z}/17^{30}\mathbb{Z}$), seven give us zero. That is, for $\{\phi_1, \phi_2, \phi_3\} \in \{\{f,g,h\}, \{f,g,h_3\}, \{f,h,h_3\}, \{f,h_2,h_3\}, \{g,h,h_2\}, \{g,h_2,h_3\}, \{h,h_2,h_3\}\}$ and $\gamma \in \{\alpha, \beta\}$, we have $\ell_{\phi_1\phi_2\phi_3,\gamma} = 0$. The non-zero values are the ones involving $\{f,g,h_2\}$, $\{f,h,h_2\}$ and $\{g,h,h_3\}$. We compute

$$\begin{aligned} (f,g,h_2)_p/\Omega_f &= -1023342994315815801374020643871 \cdot 17^2 \bmod 17^{30}, \\ (f,h,h_2)_p/\Omega_f &= 683621516993007102780000634 32 \cdot 17^2 \bmod 17^{30}, \\ (h_2,f,g)_p/\Omega_{h_2} &= -2631698743570631185431705415466 \cdot 17^2 \bmod 17^{30}, \end{aligned}$$

$$(h_2, f, h)_p/\Omega_{h_2} = 2485472478307405997935406477737 \cdot 17^2 \bmod 17^{30}.$$

Thus,

$$\frac{(f,g,h_2)_p}{(f,h,h_2)_p} \Big/ \frac{(h_2,f,g)_p}{(h_2,f,h)_p} = 1 \bmod 17^{25}.$$

Now that we have seen how to get around the issue of computing the periods, we show that our algorithms, thanks to Theorem 4.8, allow us to recover the value of periods Ω_ϕ for modular forms ϕ of weight greater than 2, as is described in Section 5.1.

Example 3. Fix again $p := 17$. Let $f, g, h_2, h_3 \in S_4(\mathbb{Q}, 45)$ be the same as in Example 2 and let $f_0 \in S_2(\mathbb{Q}, 45)$ be the newform given by $f_0 = q + q^2 - q^4 - q^5 - 3q^8 - q^{10} + \ldots$. We compute

$$\begin{aligned}
(f_0, f, h_3)_p/\Omega_{f_0} &= 16513223984800935050336063815246 \cdot 17^3 \bmod 17^{30},\\
(f, h_3, f_0)_p/\Omega_f &= 13539421372161396100812664727177 \cdot 17 \bmod 17^{30},\\
(f_0, h_2, g)_p/\Omega_{f_0} &= -3366884595101012754561302551722 \cdot 17^2 \bmod 17^{30},\\
(h_2, g, f_0)_p/\Omega_{h_2} &= 93393936291523115360189136554 \bmod 17^{30}.
\end{aligned}$$

Using Kedlaya's algorithm, we also compute

$$\Omega_{f_0} = \langle \omega_{f_0}, \phi(\omega_{f_0}) \rangle = 73740522216959426358743952636082111 \cdot 17 \bmod 17^{30}.$$

Thus, we deduce that we must have

$$\begin{aligned}
\Omega_f = \Omega_{f_0} \cdot \frac{(f_0, f, h_3)_p/\Omega_{f_0}}{(f, h_3, f_0)_p/\Omega_f} &= -8862546113964214628352195959100 \cdot 17^3 \bmod 17^{27},\\
\Omega_{h_2} = \Omega_{f_0} \cdot \frac{(f_0, h_2, g)_p/\Omega_{f_0}}{(h_2, g, f_0)_p/\Omega_{h_2}} &= -1728830956772474294735820116226 \cdot 17^3 \bmod 17^{26}.
\end{aligned}$$

Example 4. Thanks to Example 3, we have computed Ω_f and Ω_{h_2}. We can thus go back to Example 2 and calculate

$$\begin{aligned}
(f, g, h_2)_p &= -239652798828174535366407660241 \cdot 17^5 \bmod 17^{30},\\
(h_2, f, g)_p &= 5530974613520227843573162330816 \cdot 17^5 \bmod 17^{30},\\
(f, h, h_2)_p &= -853772346178158460670635373010 \cdot 17^5 \bmod 17^{30},\\
(h_2, f, h)_p &= -853772346178158460670635373010 \cdot 17^5 \bmod 17^{30}.
\end{aligned}$$

And we have $(f, g, h_2)_p = (h_2, f, g)_p \bmod 17^{30}$ and $(f, h, h_2)_p = (h_2, f, h)_p \bmod 17^{30}$.

We conclude this section with two longer examples involving different modular forms of different weights.

Example 5. Fix $p = 11$ and let $f_0 \in S_2(\mathbb{Q}, 21)$ and $f, g, h \in S_6(\mathbb{Q}, 21)$ be the cuspidal newforms given by

$$\begin{aligned}
f_0 &= q - q^2 + q^3 - q^4 - 2q^5 - q^6 - q^7 + 3q^8 + q^9 + 2q^{10} + 4q^{11} + \ldots,\\
f &= q + q^2 - 9q^3 - 31q^4 - 34q^5 - 9q^6 - 49q^7 - 63q^8 + 81q^9 - 34q^{10} - 340q^{11} + \ldots,\\
g &= q + 5q^2 + 9q^3 - 7q^4 + 94q^5 + 45q^6 - 49q^7 - 195q^8 + 81q^9 + 470q^{10} + 52q^{11} + \ldots,\\
h &= q + 10q^2 + 9q^3 + 68q^4 - 106q^5 + 90q^6 - 49q^7 + 360q^8 + 81q^9 - 1060q^{10} + 92q^{11} + \ldots.
\end{aligned}$$

From Kedlaya's algorithm, we have

$$\Omega_{f_0} = 412797842384875685536202567431940950593928402977097 \cdot 11 \bmod 11^{50}.$$

Consider the triple (f_0, f, g). We can compute

$$(f_0, f, g)_p/\Omega_{f_0} = -22575994543261422392767590042668891528437554 60 \cdot 11^5 \bmod 11^{49},$$
$$(f, g, f_0)_p/\Omega_f = -2816145142524823359002534585019971120441513443 \bmod 11^{44},$$
$$(g, f_0, f)_p/\Omega_g = -1202790078682800562850336220378526707376378726 \bmod 11^{44}.$$

This allows us to recover the periods:

$$\begin{aligned} \Omega_f &= \Omega_{f_0} \cdot \frac{(f_0, f, g)_p/\Omega_{f_0}}{(f, g, f_0)_p/\Omega_f} = -2509689183927003985676644860386486830080817519 \cdot 11^6 \bmod 11^{50}, \\ \Omega_g &= \Omega_{f_0} \cdot \frac{(f_0, f, g)_p/\Omega_{f_0}}{(g, f_0, f)_p/\Omega_g} = 2597224237884861326788056615405141084095558737 \cdot 11^6 \bmod 11^{50}. \end{aligned} \tag{28}$$

Consider now the triple (f_0, f, h). We can compute

$$(f_0, f, h)_p/\Omega_{f_0} = -2847504000645971661684808020815460021295815552 \cdot 11^4 \bmod 11^{50},$$
$$(f, h, f_0)_p/\Omega_f = 2088611347860598644979938539972864115298780 26 \cdot 11^{-1} \bmod 11^{50},$$
$$(h, f_0, f)_p/\Omega_h = 1505623403185356560351173050853572436950394 36 \bmod 11^{50}.$$

This allows us to recover the periods:

$$\begin{aligned} \Omega_f &= \Omega_{f_0} \cdot \frac{(f_0, f, h)_p/\Omega_{f_0}}{(f, h, f_0)_p/\Omega_f} = -9342144975987991033136363768117250286649236 38 \cdot 11^6 \bmod 11^{50}, \\ \Omega_h &= \Omega_{f_0} \cdot \frac{(f_0, f, h)_p/\Omega_{f_0}}{(h, f_0, f)_p/\Omega_h} = 1135142804419315201548085509390534816579616824 \cdot 11^5 \bmod 11^{49}. \end{aligned} \tag{29}$$

Note that we can also check that the two values we obtained for the period Ω_f in Equations (28) and (29) agree modulo 11^{46}. We can also compute

$$(f, g, h)_p/\Omega_f = 4026898582228757695797748499825182997898680 4 \cdot 11^2 \bmod 11^{49},$$
$$(g, h, f)_p/\Omega_g = -5234141898767450291310309034252597686927946 0 \cdot 11^2 \bmod 11^{49},$$
$$(h, f, g)_p/\Omega_h = -5183291164097188740186258999820123155166328 4 \cdot 11^3 \bmod 11^{50}.$$

This finally allows us to calculate, using Equations (28) and (29), the full values:

$$(f, g, h)_p = 2098691758998671846919428710727628689530731 1 \cdot 11^8 \bmod 11^{50},$$
$$(g, h, f)_p = -2291456031114395478251838824657372595655758 6 \cdot 11^8 \bmod 11^{50},$$
$$(h, f, g)_p = 7861733475215692445486373857156179960213682 \cdot 11^8 \bmod 11^{50}.$$

And we can check that all these values agree modulo 11^{48}.

Example 6. Fix again $p = 11$. Let $f_0 \in S_2(\mathbb{Q}, 26)$, $f, g, h \in S_4(\mathbb{Q}, 26)$ and $f_1, f_2, f_3 \in S_8(\mathbb{Q}, 26)$ be the cuspidal newforms given by

$$f_0 = q - q^2 + q^3 + q^4 - 3q^5 - q^6 - q^7 - q^8 - 2q^9 + 3q^{10} + 6q^{11} + ...,$$
$$f_1 = q + 8q^2 - 27q^3 + 64q^4 - 245q^5 - 216q^6 - 587q^7 + 512q^8 - 1458q^9 - 1960q^{10} - 3874q^{11} + ...,$$
$$f_2 = q + 8q^2 - 87q^3 + 64q^4 + 321q^5 - 696q^6 - 181q^7 + 512q^8 + 5382q^9 + 2568q^{10} + 7782q^{11} + ...,$$
$$f_3 = q - 8q^2 - 39q^3 + 64q^4 + 385q^5 + 312q^6 - 293q^7 - 512q^8 - 666q^9 - 3080q^{10} - 5402q^{11} + ...,$$
$$f = q + 2q^2 - q^3 + 4q^4 + 17q^5 - 2q^6 - 35q^7 + 8q^8 - 26q^9 + 34q^{10} + 2q^{11} + ...,$$
$$g = q + 2q^2 + 4q^3 + 4q^4 - 18q^5 + 8q^6 + 20q^7 + 8q^8 - 11q^9 - 36q^{10} - 48q^{11} + ...,$$
$$h = q - 2q^2 + 3q^3 + 4q^4 + 11q^5 - 6q^6 + 19q^7 - 8q^8 - 18q^9 - 22q^{10} - 38q^{11} + ...$$

From Kedlaya's algorithm, we have

$$\Omega_{f_0} = 390581636402185053366232716528660201295552925543487 \cdot 11 \bmod 11^{50}.$$

Consider the triple (f_0, f_1, f_2). We can compute

$$\begin{aligned}
(f_0, f_1, f_2)_p/\Omega_{f_0} &= -5933660141750195368504774740219722366045619600 \cdot 11^7 \bmod 11^{50}, \\
(f_1, f_2, f_0)_p/\Omega_{f_1} &= 14109208854192176214141915814693455702656065 \cdot 11 \bmod 11^{50}, \\
(f_2, f_0, f_1)_p/\Omega_{f_2} &= -77937947487847815992579716749595754463507 26 \cdot 11 \bmod 11^{50}.
\end{aligned}$$

This allows us to recover the periods:

$$\begin{aligned}
\Omega_{f_1} &= \Omega_{f_0} \cdot \frac{(f_0, f_1, f_2)_p/\Omega_{f_0}}{(f_1, f_2, f_0)_p/\Omega_{f_1}} = -21027051765176602834841561415436233070939 2521 \cdot 11^7 \bmod 11^{50}, \\
\Omega_{f_2} &= \Omega_{f_0} \cdot \frac{(f_0, f_1, f_2)_p/\Omega_{f_0}}{(f_2, f_0, f_1)_p/\Omega_{f_2}} = -2888149427215932149679133489787228786495075 78 \cdot 11^7 \bmod 11^{50}.
\end{aligned}$$

Now in order to recover $\Omega_f, \Omega_g, \Omega_h$, we compute

$$\begin{aligned}
(f, f_1, f_3)_p/\Omega_f &= 40903568201933522569570898222005174659773400 \cdot 11^6 \bmod 11^{47}, \\
(f_1, f_3, f)_p/\Omega_{f_1} &= -1371650302863648283749356335039702487573085 \cdot 11^2 \bmod 11^{43}, \\
(g, f_1, f_3)_p/\Omega_g &= 22042094529555547504357714056538546000021128 0 \cdot 11^5 \bmod 11^{46} \\
(f_1, f_3, g)_p/\Omega_{f_1} &= 1458224252254476116040209429849988597407090 \cdot 11^2 \bmod 11^{43}, \\
(h, f_2, f_2)_p/\Omega_h &= -22167932026142135533189834503070255673967600 \cdot 11^6 \bmod 11^{47} \\
(f_2, f_2, h)_p/\Omega_{f_2} &= -1179453771945534511715867212869271933099333 \cdot 11^2 \bmod 11^{43}.
\end{aligned}$$

This allows us to recover the periods:

$$\begin{aligned}
\Omega_f &= \Omega_{f_1} \cdot \frac{(f_1, f_3, f)_p/\Omega_{f_1}}{(f, f_1, f_3)_p/\Omega_f} = -89977488745000891823159385117660744807295 8 \cdot 11^3 \bmod 11^{44}, \\
\Omega_g &= \Omega_{f_1} \cdot \frac{(f_1, f_3, g)_p/\Omega_{f_1}}{(g, f_1, f_3)_p/\Omega_g} = 36578899966340566317653585313947952362533 \cdot 11^4 \bmod 11^{45}, \\
\Omega_h &= \Omega_{f_2} \cdot \frac{(f_2, f_2, h)_p/\Omega_{f_2}}{(h, f_2, f_2)_p/\Omega_h} = -177895636429556192548799527236171497021933 9 \cdot 11^3 \bmod 11^{44}.
\end{aligned}$$

We finally can calculate the full values:

$$\begin{aligned}
(f, g, h)_p &= 4793591678573896487795934783533995778910 20 \cdot 11^5 \bmod 11^{46}, \\
(g, h, f)_p &= 1399506016598818090453046501872791514634546 \cdot 11^5 \bmod 11^{46}, \\
(h, f, g)_p &= 2095226804671605448791510983070380539977212 \cdot 11^5 \bmod 11^{46}.
\end{aligned}$$

And we can check that all these values agree modulo 11^{43}.

5.3. **Failure of symmetry for odd weights.** Let f, g, h be modular forms of balanced weights k, ℓ, m such that k is even and ℓ, m are odd. Theorem 4.8 tells us that $(f, g, h)_p = (g, h, f)_p = -(h, f, g)_p$, and we thus see that we do not have perfect symmetry because of the factor -1 appearing in the last term. In fact, the only way to have this perfect symmetry in general is for $(f, g, h)_p$ to be trivial when its inputs do not all have even weights.

Experimental evidence shows that this is not actually the case and thus implies that we cannot expect $(f, g, h)_p$ to always be symmetric (when the inputs are permuted cyclically), if the weights are not all even. We present our examples below.

Example 7. Let χ be the Legendre symbol $\left(\frac{\cdot}{11}\right)$. Let $f_0 \in S_2(\mathbb{Q}, \Gamma_0(11))$ and $f \in S_7(\mathbb{Q}, \Gamma_1(11), \chi)$ be the cuspidal newforms given by

$$\begin{aligned} f_0 &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} + ..., \\ f &= q + 10q^3 + 64q^4 + 74q^5 - 629q^9 - 1331q^{11} + ... \end{aligned}$$

Pick $p = 23$. We have $a_p(f_0) \cdot a_p(f) \neq 0$. Using our algorithm, we calculate $(f_0, f, f)_p$ to be

$$315469253629851924796271834643122058215784315218697403223554 \cdot 23^7 \mod 23^{50}.$$

In particular, $(f_0, f, f)_p \neq 0$.

Example 8. Let χ be the Legendre symbol $\left(\frac{\cdot}{11}\right)$. Let $f_0 \in S_2(\mathbb{Q}, \Gamma_0(11))$ and $f \in S_5(\mathbb{Q}, \Gamma_1(11), \chi)$ be the cuspidal newforms given by

$$\begin{aligned} f_0 &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} + ..., \\ f &= q + 7q^3 + 16q^4 - 49q^5 - 32q^9 + 121q^{11} + ... \end{aligned}$$

Pick $p = 23$. We have $a_p(f_0) \cdot a_p(f) \neq 0$. Using our algorithm, we calculate $(f_0, f, f)_p$ to be

$$650771328793699905211695160571448949243473028954130187789476 4 \cdot 23^5 \mod 23^{50},$$

which is non-zero.

Example 9. Let $f \in S_2(\mathbb{Q}, \Gamma_0(15))$, $g, h \in S_3(\mathbb{Q}, \Gamma_1(15))$ be the cuspidal newforms given by

$$\begin{aligned} f &= q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + q^9 - q^{10} - 4q^{11} + ..., \\ g &= q + q^2 - 3q^3 - 3q^4 + 5q^5 - 3q^6 - 7q^8 + 9q^9 + 5q^{10} + ..., \\ h &= q - q^2 + 3q^3 - 3q^4 - 5q^5 - 3q^6 + 7q^8 + 9q^9 + 5q^{10} + ... \end{aligned}$$

Pick $p = 13$. Note that we actually have $a_p(f) \neq 0$ but $a_p(g) = a_p(h) = 0$ here. This doesn't pose any issues to our algorithms. We obtain

$$(f, g, h)_p = (f, h, g)_p = 5764075789663490161187104440523013115635612942518564 9 \cdot 13 \mod 13^{48}.$$

In particular, $(f, g, h)_p \neq 0$ and is symmetric in the 2nd and 3rd variables.

Example 10. Let $f \in S_2(\mathbb{Q}, \Gamma_0(15))$, $g, h \in S_5(\mathbb{Q}, \Gamma_1(15))$ be the cuspidal newforms given by

$$\begin{aligned} f &= q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + q^9 - q^{10} - 4q^{11} + ..., \\ g &= q + 7q^2 - 9q^3 + 33q^4 - 25q^5 - 63q^6 + 119q^8 + 81q^9 - 175q^{10} + ..., \\ h &= q - 7q^2 + 9q^3 + 33q^4 + 25q^5 - 63q^6 - 119q^8 + 81q^9 - 175q^{10} + ... \end{aligned}$$

Pick $p = 17$. We have $a_p(f) \cdot a_p(g) \cdot a_p(h) \neq 0$. We then calculate:

$$(f, g, h)_p = (f, h, g)_p = 896030842534926858461272575207658231678111308389785838 0 \cdot 17^5 \mod 17^{50} \neq 0.$$

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