Computational Aspects of Modular Forms and a p-adic Triple Symbol

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A thesis submitted for the degree of
Doctor of Philosophy
Trinity 2023

Abstract

In this thesis we study a p-adic symbol for triples of modular forms which was proposed to the author by Henri Darmon. Our main achievements are as follows. We prove the various symmetry properties of this p-adic triple product. We develop and successfully implement an efficient algorithm for calculating it; in the case in which all the forms have weight greater than two we require an auxiliary non-vanishing hypothesis. And we illustrate the application of our algorithm with numerous examples. A curious consequence of our work, relating to our non-vanishing hypothesis, is an efficient method to calculate certain Poincare pairings in higher weight.

Our symbol is intimately related to p-adic L-functions for triples of modular forms. However, we do not study at all continuity properties of our p-adic triple symbol. Indeed, any well-behaved variation of our symbol in the first variable is likely to be extremely subtle to study, and there may well be no way at all of making sense of this.

Acknowledgments

I would first and foremost like to thank my supervisor Professor Alan Lauder. He has always been very kind and encouraging, and helped make me the independent researcher I am today.

I am also grateful to the other professors that had a valuable input to my thesis, most notably Professors Henri Darmon, David Loeffler, and Victor Rotger. Their insight, advice and patience was invaluable and allowed me to complement my computational work with the appropriate theoretical results. The many conversations we had, in Montreal, Warwick, Zurich, and Milan, during the second half of my DPhil, were crucial to allowing me to complete my thesis as well as a paper containing the main results of my thesis.

I also would like to mention my collaborators Federico Pintore and Mattia Veroni. Although our collaborations only dealt with research in cryptography, I am thankful for their support and friendship. Working with them truly was a highlight of the past few years. I look forward for more to come.

In addition to the actual work we did together, I learned so much from them about the process of effectively conducting collaborations, research and the submission of preprints to journals. Federico, in particular, gave me a lot of practical advice while submitting my first preprint.

I also need to thank my colleagues and friends at Oxford that have been of great support, especially during the tougher terms. There would be too many of them to name here, in addition to the fear of forgetting to mention them all. Some I have met at the Mathematical Institute, some I have met through friends and other colleagues, and so many I have met at Wadham College, in the rowing club, the MCR, or the dining hall. To all these friends, from the bottom of my heart, I say: *thank you for being such amazing friends, I really hope to stay in touch and see you all very soon, somewhere.*

I am extremely grateful for the McCall MacBain Foundation for their partial funding of my DPhil at Wadham College, at the University of Oxford. Their contribution allowed me to pursue this degree, a truly life changing experience that I will forever treasure. I am also specifically thankful to John McCall MacBain for his personal interest in my success in this endeavor. His genuine care and support have been a source of inspiration and I greatly value the human connection we established.

Finally, and most importantly, I would like to (and need to) thank my family for their everlasting love, care, support and advice. Their presence in my life alleviated the difficulties and hardships associated to graduate studies. I dedicate this work to them, as this would not have been possible without their everlasting love and support. Thank you to my parents Tanios and Reine, to my brothers Ziad and Rayan and to my sister Sarah.

Contents

Chapter 1

Introduction

1.1 A simple example

We begin by introducing our triple symbol in the simplest setting. All the notation used here will be fully defined in the body of this thesis.

Let f, g, h be three cuspidal eigenforms over $\mathbb{Q}$ of weight 2, level N and trivial characters. Fix a prime p and assume that $p \nmid N$. Let α_f and β_f be the roots of the *Hecke polynomial*

$$x^2 - a_p(f)x + p.$$

Assume that the modular form f is *regular at* p, i.e. that α_f and β_f are different. Assume as well that f is *ordinary at* p, i.e. that one of the roots of $x^2 - a_p(f)x + p$, say α_f, is a p-adic unit. Define the following two modular forms:

$$\begin{aligned} f_\alpha(q) &:= f(q) - \beta_f f(q^p); \\ f_\beta(q) &:= f(q) - \alpha_f f(q^p). \end{aligned}$$

We call f_α and f_β the *p-stabilizations* of f. They have level pN, and are eigenforms for the U_p operator with respective eigenvalues α_f and β_f. Since we assumed that α_f is a unit, it is customary to call f_α the *ordinary p-stabilization* of f. Define the following Euler factors:

$$\begin{aligned} \mathcal{E}(f,g,h) &:= (1-\beta_f\alpha_g\alpha_h p^{-2})(1-\beta_f\alpha_g\beta_h p^{-2})(1-\beta_f\beta_g\alpha_h p^{-2})(1-\beta_f\beta_g\beta_h p^{-2}); \\ \tilde{\mathcal{E}}(f,g,h) &:= (1-\alpha_f\alpha_g\alpha_h p^{-2})(1-\alpha_f\alpha_g\beta_h p^{-2})(1-\alpha_f\beta_g\alpha_h p^{-2})(1-\alpha_f\beta_g\beta_h p^{-2}); \\ \mathcal{E}_0(f) &:= 1-\beta_f^2 p^{-1}; \qquad \tilde{\mathcal{E}}_0(f) := 1-\alpha_f^2 p^{-1}; \\ \mathcal{E}_1(f) &:= 1-\beta_f^2 p^{-2}; \qquad \tilde{\mathcal{E}}_1(f) := 1-\alpha_f^2 p^{-2}. \end{aligned} \tag{1.1}$$

Let λ_{f_γ} be the projection over f_γ; it is the unique Hecke-equivariant linear functional that factors through the Hecke eigenspace associated to f_γ and is normalized to send f_γ to 1 (cf. Definition 2.7 in [Loe18]). Let $\mathrm{d} := q\frac{\mathrm{d}}{\mathrm{d}q}$ be the Serre differential operator and $\omega_f := f(q)\frac{dq}{q}$ the differential associated to f. Consider the quantity

$$\frac{\langle \omega_f, \phi(\omega_f)\rangle}{p}\left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)}\,\beta_f\,\lambda_{f_\alpha}\left(\mathrm{d}^{-1}(g^{[p]})\times h\right) + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)}\,\alpha_f\,\lambda_{f_\beta}\left(\mathrm{d}^{-1}(g^{[p]})\times h\right)\right), \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ is the Poincaré pairing and ϕ is the Frobenius map. It turns out that this quantity is independent – up to a sign – of the order of f, g and h. This result is particularly surprising since the quantity in (1.2) does not appear to be symbolically symmetric in f, g and h. This will fit into the framework of this thesis, as we relate this quantity to the image of certain diagonal cycles under the p-adic Abel-Jacobi map.

The above can even be generalized to modular forms of higher weight and any characters satisfying $\chi_f\chi_g\chi_h = 1$, which we will do in Section 4.2. In that case, one needs to adjust the Euler factors from (1.1) and introduce an extra factor and some twists by χ_f^{-1} in (1.2). One would also require that the weights be balanced, i.e. that the largest one is strictly smaller than the sum of the other two.

In order to explicitly calculate (1.2), for modular forms of general weight, we need certain computational tools, namely being able to compute ordinary projections of nearly overconvergent modular forms, as well as projection over the slope α subspace for α not necessarily zero. In [Lau14] (see also [Lau11]), the author describes an algorithm allowing the calculation of ordinary projections of overconvergent modular forms. We introduce here improvements to this algorithm, allowing us to accomplish the aforementioned tasks. The use of this new algorithm is not restricted to this work. The experimental calculations detailed in Chapter 5, on the symmetry of (1.2), provide additional support to the fact that our algorithm is functioning properly.

An additional application of our code is the calculation of certain periods of modular forms. Indeed, using the symmetry of our new p-adic triple symbol, introduced in Section 4.2, we explain how one can use our algorithms to compute the Poincaré pairing $\Omega_f := \langle\omega_f, \phi(\omega_f)\rangle$, where ϕ denotes the Frobenius action and f is a newform of any weight. See [DL21], [DLR16] and Section III.5 of [Nik11] for instances where this pairing appears in the literature. There are currently no known ways of evaluating general Poincaré pairings, and the value of Ω_f has so far only been computed in cases where f has weight 2 using Kedlaya's algorithm [Ked01].

1.2 Structure of the thesis

We now describe the structure of the thesis. It is divided as follows.

In Chapter 2, we find it relevant to mention certain theoretical aspects of (classical) modular forms. Indeed, one of the best ways of studying p-adic modular forms is to relate them to classical modular forms. We thus describe the various ways one can view them. We also discuss the U_p operator and the slope decomposition it induces.

In Chapter 3, we recall a known algorithm to compute projections of modular forms, as our ultimate goal is to compute certain p-adic triple symbols. We expand on the known algorithm and generalize it. Some of the methods that we will describe are due to David Loeffler. In particular, the approaches used in Sections 3.1.3 and 3.2 were suggested by him.

In Chapter 4, we recall the Rankin-Garrett triple product p-adic L-function. We then, inspired by it, define a p-adic symbol for triples of modular forms. In the remainder of the chapter, we study the symmetry properties of our new p-adic symbol, both when the first variable is fixed and when all three inputs are allowed to vary. We also explain why our symbol cannot satisfy full symmetry in the case of odd weights.

Chapter 5 is dedicated to presenting our numerical results, which support both our formulas and the well-functioning of our algorithms. We make sure to include varied examples; in particular we include examples of overconvergent modular forms as well as nearly overconvergent modular forms. We also include forms of non-trivial character and odd weight.

In Chapter 6, we present a curious application of our algorithms. Indeed, we can exploit the symmetry result for our p-adic triple symbol to compute Poincaré pairings of the form $\langle \omega_f, \phi(\omega_f) \rangle$. Such pairings are p-adic analogs of the Petersson norm, and have so far only been computed in the case where f is a cuspidal newform of weight 2 over $\mathbb{Q}$. We show how to compute them in the case where f can have any weight.

In Chapter 7, we discuss some aspects of research in computational number theory that are often not mentioned in research papers. Indeed, in this section of the thesis, we take the time to discuss some issues that arose at the interface of experimental calculations and theoretical research. We explain how experimental calculations can be a valuable tool to help guide and confirm theoretical beliefs, by giving examples from the author's own experience while conducting the research presented in this thesis.

Chapter 2

Preliminaries

2.1 Modular forms

2.1.1 Classical Modular forms

Modular forms are usually defined in the following way (cf. [DS05, DI95, Dar04]), which lends itself well to computations. To this end, it is convenient to introduce the slash notation for a complex valued function f on the upper half plane $\mathcal{H}$:

$$(f|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma\tau), \tag{2.1}$$

for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$, where k is an integer.

The factor of $(c\tau + d)^{-k}$ appearing in (2.1) is called a factor of automorphy. We shall denote it by $j_k(\gamma, \tau)$. It is a holomorphic function from $\mathrm{SL}_2(\mathbb{Z}) \times \mathcal{H}$ to $\mathbb{C}^\times$, satisfying the *cocycle relation*:

$$j_k(\gamma_1\gamma_2, \tau) = j_k(\gamma_1, \gamma_2\tau) j_k(\gamma_2, \tau).$$

We are now ready to define modular forms of weight k and level Γ, where Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1.1 (Modular forms, Version 1)**.** A modular form f of weight k and level Γ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that $(f|_k\gamma)(\tau) = f(\tau)$ for all $\gamma \in \Gamma$ and all $\tau \in \mathcal{H}$. Moreover, we require that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, there exists some $h \in \mathbb{N}$ such that $f|_k\gamma$ has a Fourier expansion

$$(f|_k\gamma)(\tau) = \sum_{n \geq 0} a_n(\gamma)(q^{1/h})^n,$$

where $q = e^{2\pi i\tau}$.

In practice, the main levels $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ that one considers are

$$\Gamma(N) := \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) : M \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod N \right\},$$

$$\Gamma_1(N) := \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) : M \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \mod N \right\},$$
$$\Gamma_0(N) := \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) : M \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \mod N \right\},$$

for $N \in \mathbb{N}$. The subgroups $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ are singled out as they are directly related to the reduction map

$$\mathrm{SL}_2(\mathbb{Z}) \xrightarrow{\text{mod } N} \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Indeed, $\Gamma(N)$ arises as the kernel of the above reduction map, $\Gamma_0(N)$ is the preimage of the Borel subgroup (i.e. the subgroup of upper triangular matrices) and $\Gamma_1(N)$ is the preimage of the subgroup of upper triangular matrices with 1s on the diagonal. We also see in Remark 1 that these groups have specific roles and can be interpreted in certain meaningful ways. We will therefore always assume that the level Γ of a modular form is one of $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$, for $N \in \mathbb{N}$.

Definition 2.1.1 has the advantage of being direct, simple and easy to use for computations. However, it is not the best conceptual way to view modular functions, especially, when trying to prove theoretical results about them, or generalize their construction to arbitrary number fields.

Indeed, let us introduce the concept of level Γ structure (see Chapter 3 in [KM85] for more on this). This will encode the interaction between the subgroup Γ and the modular forms. Then, we'll be able to define modular forms in a more conceptual way.

Definition 2.1.2. Let E be an elliptic curve over some base ring R (or over a base scheme S, if we want to be more general) and let Γ be one of $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$ for $N \in \mathbb{N}$ which is invertible in R. We say that α is a level Γ structure on E if it is

- an isomorphism of R-group schemes $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N]$, if $\Gamma = \Gamma(N)$;
- an injective homomorphism $\alpha : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]$, if $\Gamma = \Gamma_1(N)$;
- a cyclic isogeny $\alpha : E \longrightarrow E'$ of degree N, to some elliptic curve E', if $\Gamma = \Gamma_0(N)$.

In the literature, level $\Gamma(N)$ structures, level $\Gamma_1(N)$ structures and level $\Gamma_0(N)$ structures are sometimes refer to as *full level N structures*, *arithmetic level N structures* and *Borel level N structures*, respectively (see [KM85] and [Gou88]).

Remark 1 (See pages 37-38 in [DS05]). A full level $\Gamma(N)$ structure is equivalent to giving a (Drinfeld) basis (P, Q) for $E[N]$, i.e. a pair (P, Q) that generates $E[N]$ and has Weil pairing $e_N(P, Q) = e^{2\pi i/N}$. An arithmetic level $\Gamma_1(N)$ structure is equivalent to giving a fixed point of exact order N on E. Finally, a level $\Gamma_0(N)$ structure is equivalent to giving a cyclic subgroup of E order N. See also Theorem 1.5.1 in [DS05] for more.

Recall now, from the theory of elliptic curves, the fact that isomorphism classes of elliptic curves over $\mathbb{C}$ are classified by homothety classes of lattices, which in turn are also classified by the complex upper half plane modulo the action of integer linear fractional transformations. This gives the following bijections:

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} \xleftrightarrow{\sim} \{\text{lattices } \Lambda \subseteq \mathbb{C} \text{ up to homothety}\}$$

$$\xrightarrow{\sim} \{\text{elliptic curves } E/\mathbb{C} \text{ up to isomorphism}\}.$$

Moreover, if we look at lattices in $\mathbb{C}$, *not* up to homothety, it turns out that they also classify elliptic curves up to isomorphism with the additional structure of a non-vanishing differential. Indeed,

$$\{\text{lattices } \Lambda \subseteq \mathbb{C}\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{elliptic curves } E \text{ up to isomorphism} \\ \text{with a non-vanishing differential } \omega \end{array} \right\}.$$
$$\Lambda \longmapsto (E_\Lambda := \mathbb{C}/\Lambda = (\wp_\Lambda(z), \wp'_\Lambda(z)), \omega_\Lambda := \mathrm{d}z),$$

where

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{x \in \Lambda - \{0\}} \frac{1}{(z-x)^2} - \frac{1}{x^2}$$

is the Weirestrass $\wp$ function, allowing us to transition from lattices Λ to elliptic curves E_Λ.

Finally, with this in mind, we can look back at Definition 2.1.1 and see that the domain of modular functions can be equivalently thought of as just being the set of elliptic curves up to isomorphism with a non-vanishing differential and a certain regularity under the action of Γ (which can be encoded via level Γ structures). This helps understand where the following more abstract definition of modular forms comes from.

Definition 2.1.3 (Modular forms, Version 2)**.** A modular form f over $\mathbb{C}$ of weight k and level Γ is a holomorphic (at the cusps) complex valued function $f(E, \omega, \iota)$ on "test objects" (E, ω, ι) where E is an elliptic curve over some $\mathbb{C}$-algebra A, ω is a non vanishing differential on E, and ι is level Γ structure on E, such that $f(E, \omega, \iota)$

(i) only depends on the isomorphism class of (E, ω, ι);

(ii) commutes with arbitrary change of base (of the field/ring on which E is defined);

(iii) is homogeneous of degree $-k$ in its second variable, i.e. $f(E, \lambda\omega, \iota) = \lambda^{-k} f(E, \omega, \iota)$.

Remark 2. To be completely rigorous, we need to point out that given a test object (E, ω, ι) and an isomorphism of elliptic curves $\rho : E \longrightarrow E'$, we get a new test object (E', ω', ι'), where ω' is the pushforward of ω through ρ and ι' is obtained by composing ι with ρ (or $\rho^\vee$) in an appropriate way, depending on which $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ we are dealing with.

The advantage of this definition is that it easily generalizes to any base scheme S, rather than always taking $\mathbb{C}$. This is the approach taken by Katz when he introduced overconvergent modular forms. We will see this in Definition 2.1.5 of Section 2.1.2.

Up until now, we have see modular forms as functions on $\mathcal{H}$ that transform nicely when acted upon by $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. We can alternatively view them as functions on $\mathcal{H}/\Gamma$, but with some more structure to compensate for the fact that their domain is now $\mathcal{H}/\Gamma$ instead of $\mathcal{H}$. This gives us this last definition, where we view modular forms as being sections of line bundles. There will be complications when dealing with the case $\Gamma = \Gamma_0(N)$. Indeed, $\Gamma_0(N)$ is not a torsion free subgroup of $\mathrm{SL}_2(\mathbb{Z})$, as it contains the matrix $-\mathrm{Id}$ for all N. This will cause issues when considering universal modular curves.

In contrast, the curves obtained from compacting $\mathcal{H}/\Gamma_1(N)$ and $\mathcal{H}/\Gamma(N)$ already come with a universal modular curve (at least for $N > 4$ and $N \geq 3$ respectively).

First, we define the *Hodge line bundle* $\mathscr{L}_\Gamma^k$ on $\Gamma\backslash\mathcal{H}$ as follows. Recall that all line bundles on $\mathbb{C}$ must be trivial because $\mathbb{C}$ is contractible. We can then take the trivial line bundle $\mathcal{H}\times\mathbb{C} \to \mathcal{H}$ on $\mathcal{H}$ and reduce it modulo Γ via the action $(\tau, t) \mapsto (\gamma \star \tau, t \cdot j_k(\gamma, \tau))$, as in Figure 2.1. This line bundle is completely determined by the factor of automorphy $j_k(\gamma, \tau)$, which in turn depends precisely on the weight k and the level Γ.

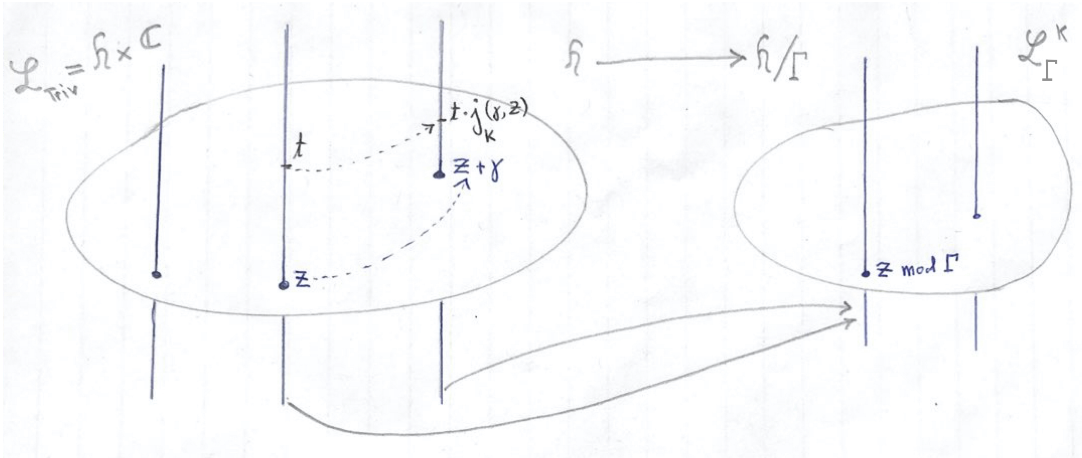

Figure 2.1: The Hodge line bundle $\mathscr{L}_\Gamma^k$ on $\Gamma\backslash\mathcal{H}$.

Let $\mathcal{H}^* := \mathcal{H} \sqcup \mathbb{P}^1(\mathbb{Q})$ and consider the compactification $\Gamma\backslash\mathcal{H}^*$ of $\Gamma\backslash\mathcal{H}$. We are now ready to view modular forms of weight k and level Γ as being *global sections of the Hodge line bundle* $\mathscr{L}_\Gamma^k$ over $\Gamma\backslash\mathcal{H}^*$, since one can extend the Hodge line bundle from $\Gamma\backslash\mathcal{H}$ to $\Gamma\backslash\mathcal{H}^*$ when Γ is sufficiently small (see Remark 4.5 in [Gor02]). This is equivalent to saying that they are holomorphic functions f on $\mathcal{H}$ such that $f(\gamma\tau) = j_{\mathscr{L}_\Gamma^k}(\gamma, \tau) f(\tau)$, where $j_{\mathscr{L}_\Gamma^k} = j_{\Gamma,k}$ is the factor of automorphy associated to the bundle $\mathscr{L}_\Gamma^k$.

Definition 2.1.4 (Modular forms, Version 3)**.** A modular form of weight k and level Γ is an element of $H^0\left(\mathcal{X}(\Gamma), \mathscr{L}_\Gamma^k\right)$, where $\mathcal{X}(\Gamma) := \Gamma\backslash\mathcal{H}^*$ denotes the compactification of the modular curve $\Gamma\backslash\mathcal{H}$ and $\mathscr{L}_\Gamma^k$ is the Hodge line bundle on $\mathcal{X}(\Gamma)$.

When Γ is torsion free, instead of taking the Hodge line bundle $\mathscr{L}_\Gamma^k$, we can use a more general approach based on the differentials naturally associated to the modular curve $\Gamma\backslash\mathcal{H}$. Indeed, let $\tilde{\mathcal{E}_\Gamma} := \{(\tau, x) : \tau \in \mathcal{H}, x \in \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)\}$. Then the group Γ acts on $\tilde{\mathcal{E}_\Gamma}$ via $(\tau, x) \mapsto (\gamma\tau, j_{\Gamma,k}(\gamma, \tau)^{-1}x)$. This allows us to define the universal elliptic curve $\mathcal{E}_\Gamma := \Gamma\backslash\tilde{\mathcal{E}_\Gamma}$ of level Γ. Now, let $\pi : \mathcal{E}_\Gamma \longrightarrow \Gamma\backslash\mathcal{H}$ be the natural projection and let $\omega := \pi_*\Omega_{\mathcal{E}_\Gamma}$ be the pushforward onto $\Gamma\backslash\mathcal{H}$ of the (sheaf of relative) differentials on $\mathcal{E}_\Gamma$. We finally obtain the line bundle of differentials $\omega^{\otimes k}$ on $\Gamma\backslash\mathcal{H}$. We then have $H^0\left(\mathcal{X}(\Gamma), \mathscr{L}_\Gamma^k\right) = H^0\left(\mathcal{X}(\Gamma), \omega^{\otimes k}\right)$.

Note also that the above definition makes sense because one can chose an appropriate extension of the bundle $\omega^{\otimes k}$ from $\Gamma\backslash\mathcal{H}$ to $\Gamma\backslash\mathcal{H}^*$ (see Section 4 of [Gor02] for more on this). The set of modular forms of weight k and level Γ is denoted by $M_k(\Gamma)$. Some authors also consider the set of modular forms that are meromorphic at the cusps but not necessarily holomorphic and denote it by $F_k(\Gamma)$. We will call $F_k(\Gamma)$ the set of *meromorphic* modular forms of weight k and level Γ to avoid any confusion. So, we would have

$$M_k(\Gamma) = H^0\left(\Gamma\backslash\mathcal{H}^*, \mathscr{L}_\Gamma^k\right), \qquad F_k(\Gamma) = H^0\left(\Gamma\backslash\mathcal{H}, \mathscr{L}_\Gamma^k\right).$$

Definition 2.1.1 of a modular form makes it clear how we can get a Fourier expansion out of a modular form. The other two definitions are a bit more abstract. To see how this is possible, we use a parametrization of the set of elliptic curves (which modular forms can take as inputs, see Definition 2.1.3) by complex numbers via the isomorphism

$$\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z}) \xrightarrow{\sim} \mathbb{C}^\times/q^{\mathbb{Z}},$$

where $q := e^{2\pi i\tau}$. As a consequence, we obtain a parametrization $T(q)$ giving us an elliptic curve for each $q = e^{2\pi i\tau}$. We refer to $T(q)$ as the *Tate curve* (cf. Theorem V.1.1 in [Sil94]). Then, for a modular form f as in Definition 2.1.3, we define its q-expansion (at the ∞ cusp) as

$$f(q) := f\left(T(q), \omega_{\text{can}}\right) \in \mathbb{C}[[q]].$$

The reason why $f(q)$ lies in $\mathbb{C}[[q]]$ instead of $\mathbb{C}((q))$ is that modular forms have to be holomorphic at the cusps in our definitions, so in particular they will be holomorphic at the cusp ∞.

2.1.2 Overconvergent and p-adic modular forms

Fix a prime p and a finite extension K of $\mathbb{Q}_p$ and let $B := \mathcal{O}_K$ be its ring of integers. We can actually take B to be any p-adic ring, i.e. a complete separated $\mathbb{Z}_p$-algebra with the p-adic topology. Overconvergent modular forms can be thought of as being modular forms with extra convergence conditions. We will make this idea more precise. Just like for classical modular forms (see Definitions 2.1.3 and 2.1.4), we can view them as functions on test objects (cf. [Gou88, Kat73, Kat75]) or as sections of certain line bundles. Here, *test objects* of level Γ and growth condition $r \in B$ are tuples $(E/A, \omega, \iota, Y)$, where E is an elliptic curve over some B-algebra A together with a non-vanishing differential ω and a level Γ structure ι. Also, Y will just be some element of A such that $Y \cdot E_{p-1}(E,\omega) = r$. By E_{p-1}, we mean the normalized Eisenstein series of weight $p-1$. The level structures that we will consider here will be for the subgroups $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ with $N \in \mathbb{N}$ such that $p \nmid N$.

Definition 2.1.5 (Overconvergent modular forms, Version 1)**.** A p-adic modular form f of weight k, level Γ and growth condition r is a function that is holomorphic at the cusps (see Remark 3 below) sending each objects $(E/A, \omega, \iota, Y)$ of level Γ and growth condition r to $f(E/A, \omega, \iota, Y) \in A$ such that $f(E/A, \omega, \iota, Y)$

(i) only depends on the isomorphism class of (E, ω, ι, Y);

(ii) commutes with arbitrary change of base (of the ring on which E is defined);

(iii) is homogeneous of degree $-k$ in its second variable, i.e.

$$f(E/A, \lambda\omega, \iota, Y) = \lambda^{-k} f(E/A, \omega, \iota, Y).$$

Remark 3. In the above definition, and throughout this thesis, we mention p-adic modular forms f that are *holomorphic* at the cusps. For instance, when the (usual) q-expansion of $f(q)$ lies in $B[[q]]$, instead of just $B((q))$, we say that f is holomorphic at the cusp ∞. In general, one can consider the q-expansion of f at any cusp. If every such expansion lies in $B[[q]]$, then we say that f is holomorphic at the cusps. We will explain how to get such q-expansions at the end of this subsection.

The space of p-adic modular form f of weight k, level Γ and growth condition r is denoted by $M_k^{p\text{-adic}}(B, \Gamma; r)$. We might often drop the B when it is clear which space we are working with. Usually, we will have $B := \mathbb{Z}_p$. In the case where r is not a unit, $r \notin B^\times$, we say that we have an *overconvergent modular form* of weight k, level Γ and growth condition r. We denote this space by $M_k^{\text{oc}}(B, \Gamma; r)$. We will see at the end of this section how to define the vector space of overconvergent modular forms $M_k^{\text{oc}}(K, \Gamma; r)$ defined over the field K.

We can also similarly to the last section define overconvergent modular forms as sections of line bundles. This task is a bit more subtle now. For example, if $p = 3$, we would have to take the 3rd power of $\mathscr{A}$ instead of $\mathscr{A}$ in what follows (cf. Theorem 1.8.1 in [Cal13]). Assume henceforth that $p \geq 5$, in order to simplify the following construction. We define $\mathcal{X}(\Gamma)_{\leq r}$ to be the rigid analytic space (cf. [Bos09, Bos14]), over K, that is given by the set of points x of the compactified moduli scheme (over B) of elliptic curves with a level Γ structure such that $\mathrm{ord}_p(\tilde{\mathscr{A}}(x)) \leq \mathrm{ord}_p(r)$, where $\tilde{\mathscr{A}}(x)$ is a lift of the Hasse invariant $\mathscr{A}$ at x. For simplicity, we will ignore the distinction between a rigid analytic space and its underlying set of closed points. It is know that the Hasse invariant vanishes precisely on supersingular elliptic curves. Thus, we can think of $\mathcal{X}(\Gamma)_{\leq r}$ as being just like $\mathcal{X}(\Gamma)$ but with balls of radius $|r|_p$ removed around the supersingular points. In particular, if $\mathrm{ord}_p(r) = 0$ (i.e. r is invertible) then we get the ordinary "locus" $\mathcal{X}(\Gamma)_{\text{ord}}$ composed of all the ordinary elements in $\mathcal{X}(\Gamma)$.

Just like for classical modular forms, the space $\mathcal{X}(\Gamma)_{\leq r}$ has a set of differentials ω on it. They come from (the pushforward of) the holomorphic differential forms on the universal elliptic curve over B.

Definition 2.1.6 (Overconvergent modular forms, Version 2)**.** An overconvergent modular form of weight k, level $\Gamma(N)$ and growth condition r is a section of $H^0\left(\mathcal{X}(\Gamma)_{\leq r}, \omega^{\otimes k}\right)$. So we have

$$M_k^{\text{oc}}(K, \Gamma; r) = H^0\left(\mathcal{X}(\Gamma)_{\leq r}, \omega^{\otimes k}\right).$$

One can give an analogue of the above definition for overconvergent modular forms defined over B. To do so, we would need however to define an integral version of $\mathcal{X}(\Gamma)_{\leq r}$. See, for example, the remark at the bottom of page 6 in [Gou88] for more on how to do this.

Let E_{p-1} denote the normalized Eisenstein series of weight $p-1$ (and level 1). For $N \geq 3$, $p \geq 5$ and $r \in B^\times$, we have an isomorphism (cf. Theorem 6.15 in [Gor02]) expressing them in terms of an inverse limit of classical objects

$$M_k^{\text{oc}}(B, \Gamma; r) \cong \varprojlim_n \left(H^0\left(\mathcal{X}(\Gamma)_{/\mathbb{Z}_p}, \bigoplus_{j=0}^{\infty} \omega^{k+j(p-1)}\right) \otimes_{\mathbb{Z}_p} (B/p^n B) \right) \Big/ (E_{p-1} - r),$$

where $\mathcal{X}(\Gamma)_{/\mathbb{Z}_p}$ denotes the compactified moduli scheme over $\mathbb{Z}_p$ of elliptic curves with a level Γ structure and ω denotes the differentials on $\mathcal{M}(\Gamma)$. This translates to

$$M_k^{\mathrm{oc}}(B, \Gamma; r) \cong \varprojlim_n \left(\bigoplus_{j=0}^{\infty} M_{k+j(p-1)}(B/p^n B, \Gamma) \right) \Big/ (E_{p-1} - r)\,.$$

These definitions don't help us very much in terms of computations, as they are quite abstract. Thankfully, there is a convenient and computationally-friendly way of dealing with them. We will find a "Banach" basis for $M_k^{\mathrm{oc}}(B, N; r)$, allowing us to express overconvergent modular forms as series in classical objects. Actually, we will find a "Banach" basis for $M_k^{p\text{-adic}}(B, N; r)$, i.e. r doesn't need to be non-invertible. We will thus rely on classical modular forms to build the overconvergent ones.

First, assume for simplicity that $p \geq 5$ and does not divide N. When we write $M_k(B, N)$, we mean the space of modular forms over B of weight k and arithmetic level structure N, i.e. level structure $\Gamma_1(N)$. Note also that we have

$$M_k(B, N) = M_k(\mathbb{Z}_p, N) \otimes_{\mathbb{Z}_p} B.$$

Notice that the map

$$\begin{aligned} M_{k+(i-1)(p-1)}(B, N) &\hookrightarrow M_{k+i(p-1)}(B, N) \\ f &\mapsto E_{p-1} \cdot f \end{aligned}$$

is injective but not surjective for all $i \geq 1$. It also has a finite free cokernel ([Kat73], Lemma 2.6.1), so it must split. We can then, following Gouvêa's notation (see Chapter I of [Gou88]), let $A_{k+i(p-1)}(B, N)$ be a free B-module such that

$$M_{k+i(p-1)}(B, N) = E_{p-1} \cdot M_{k+(i-1)(p-1)}(B, N) \oplus A_{k+i(p-1)}(B, N).$$

For $i = 0$, let $A_k(B, N) := M_k(B, N)$. We also have

$$A_{k+i(p-1)}(B, N) = A_{k+i(p-1)}(\mathbb{Z}_p, N) \otimes_{\mathbb{Z}_p} B.$$

We can think of $A_{k+i(p-1)}(B, N)$ as the set of modular forms of weight $k + i(p-1)$ that do not come from smaller weight forms multiplied by E_{p-1}. We notice that we can write

$$\begin{aligned} M_{k+i(p-1)}(B, N) &= E_{p-1} \cdot M_{k+(i-1)(p-1)}(B, N) \oplus A_{k+i(p-1)}(B, N) \\ &= E_{p-1} \cdot \big(E_{p-1} \cdot M_{k+(i-2)(p-1)}(B, N) \oplus A_{k+(i-1)(p-1)}(B, N)\big) \\ &\quad \oplus A_{k+i(p-1)}(B, N) \\ &\quad \vdots \\ &= \bigoplus_{a=0}^{i} E_{p-1}^{i-a} \cdot A_{k+a(p-1)}(B, N). \end{aligned}$$

We are now ready to give an equivalent definition for the space of r-overconvergent modular forms.

Definition 2.1.7 (Overconvergent modular forms, Version 3)**.** The space of overconvergent modular forms of weight k, growth condition r and level $\Gamma_1(N)$ is given by

$$M_k^{\mathrm{oc}}(N; r) := \left\{ \sum_{i=0}^{\infty} r^i \frac{b_i}{E_{p-1}^i} : b_i \in A_{k+i(p-1)}(N), \lim_{i \to \infty} b_i = 0 \right\}, \tag{2.2}$$

where by $\lim_{i\to\infty} b_i = 0$, we mean that the expansion of b_i is more and more divisible by p as i goes to infinity.

Remark 4. If we take r to be invertible in equation (2.2), we will just get $M_k^{p\text{-adic}}(N;r)$. Moreover, we can also define overconvergent modular forms with a given character χ. To do so, we simply take

$$M_k^{\text{oc}}(N,\chi;r) := \left\{ \sum_{i=0}^{\infty} r^i \frac{b_i}{E_{p-1}^i} : b_i \in A_{k+i(p-1)}(N,\chi), \lim_{i\to\infty} b_i = 0 \right\}.$$

Where $A_{k+i(p-1)}(N,\chi)$ is define analogously to $A_{k+i(p-1)}(N)$ by

$$M_{k+i(p-1)}(B,N,\chi) = E_{p-1} \cdot M_{k+(i-1)(p-1)}(B,N,\chi) \oplus A_{k+i(p-1)}(B,N,\chi).$$

An expansion for $f \in M_k^{p\text{-adic}}(B,N,\chi;r)$ of the form $f = \sum_{i=0}^{\infty} r^i \frac{b_i}{E_{p-1}^i}$ is called a *Katz expansion* and we call r the growth condition. This way of writing them shows how they can be seen as being *overconvergent*, i.e. they "converge faster than standard classical modular forms" (in the p-adic topology).

Remark 5. Some authors (see [Urb14]) might use the notation $M_k^{\text{oc}}(B,N,\chi;\alpha)$ to mean $M_k^{\text{oc}}(B,N,\chi;p^\alpha)$. This is because we don't really care much about what r actually is; we only care about its p-adic valuation. Indeed, if $r = p^\alpha \cdot c$ with $p \nmid c$, then we can write $f = \sum_{i=0}^{\infty} r^i \frac{b_i}{E_{p-1}^i}$ as $f = \sum_{i=0}^{\infty} p^{\alpha i} \frac{c \cdot b_i}{E_{p-1}^i}$. If we ever use the notation $M_k^{\text{oc}}(B,N,\chi;\alpha)$ instead of $M_k^{\text{oc}}(B,N,\chi;p^\alpha)$ here, we will make it clear.

We can also talk about the space of all overconvergent modular forms $M_k^{\text{oc}}(B,N,\chi)$ without specifying the growth condition,

$$M_k^{\text{oc}}(B,N,\chi) := \bigcup_{r \notin B^\times} M_k^{\text{oc}}(B,N,\chi;r). \tag{2.3}$$

Remember that we do not include units in the above definition because this will give us p-adic modular forms that aren't overconvergent (see Definition 2.1.5). If we did allow r to also be invertible, we would obtain the set of all p-adic modular forms over B of level $\Gamma_1(N)$ and integer weight k and any growth condition,

$$M_k^{p\text{-adic}}(B,N,\chi) := \bigcup_{r \in B} M_k^{\text{oc}}(B,N,\chi;r).$$

If $r = r_0 r_1$, we then have an inclusion

$$\begin{aligned} M_k^{\text{oc}}(B,N,\chi;r) &\hookrightarrow M_k^{\text{oc}}(B,N,\chi;r_0), \\ \sum_{i=0}^{\infty} r^i \frac{b_i}{E_{p-1}^i} &\mapsto \sum_{i=0}^{\infty} r_0^i \frac{(r_1^i b_i)}{E_{p-1}^i}. \end{aligned} \tag{2.4}$$

In particular, letting $r = 1$, or any unit, we see that the space of p-adic modular forms of growth condition 1 is equal to the space of p-adic modular forms of any growth condition, i.e. $M_k^{p\text{-adic}}(B,N,\chi;1) = M_k^{p\text{-adic}}(B,N,\chi)$. And we also see that overconvergent modular forms are also p-adic modular forms:

$$M_k^{\text{oc}}(B,N,\chi) \subseteq M_k^{p\text{-adic}}(B,N,\chi).$$

Thanks to (2.4), the union in Equation (2.3) can be rewritten more appropriately as a direct limit

$$M_k^{\text{oc}}(B,N,\chi) := \varinjlim_{\text{ord}_p(r)>0} M_k^{\text{oc}}(B,N,\chi;r).$$

In [Ser73], Serre gives a different definition of p-adic modular forms, based on limits of q-expansions of classical modular forms. To be specific, we say $f=\sum_n a_n q^n$ is the limit of the sequence $f_i=\sum_n a_n^{(i)} q^n$ if $\text{val}_p(f-f_i)\to\infty$ as $i\to\infty$, where $\text{val}_p\left(\sum_n a_n q^n\right) := \inf_n \text{val}_p(a_n)$. Note that according to this general definition p-adic modular forms need not have integer weights. Indeed, the set of weights is

$$\text{Hom}_{\text{Conts}}(\mathbb{Z}_p^\times,\mathbb{Z}_p^\times)=\text{Aut}_{\text{Conts}}(\mathbb{Z}_p^\times)\cong\mathbb{Z}_p^\times\cong\mathbb{Z}/(p-1)\mathbb{Z}\times\mathbb{Z}_p,$$

for $p\neq 2$.

Definition 2.1.8 (p-adic modular forms *à la* Serre)**.** The space of p-adic modular forms of all weights in $\text{Hom}_{\text{Conts}}(\mathbb{Z}_p^\times,\mathbb{Z}_p^\times)$, denoted by $M^{p\text{-adic}}(B,\Gamma)$, is the completion of the space of classical modular forms $\bigcup_{k\in\mathbb{Z}} M_k(B,\Gamma)$.

This definition is very simple to state. On the other hand, the space of p-adic modular forms of any weight, in $\text{Hom}_{\text{Conts}}(\mathbb{Z}_p^\times,\mathbb{Z}_p^\times)$, has a very complicated structure and is quite difficult to study. It is therefore common to restrict ones attention to overconvergent modular forms instead, i.e. to exclude the case $\text{ord}_p(r)=0$, as this will give non-overconvergent p-adic modular forms.

We now introduce the Serre operator $q\frac{\mathrm{d}}{\mathrm{d}q}$, which is often denoted by Θ,

$$\begin{aligned} q\frac{\mathrm{d}}{\mathrm{d}q} : M_k^{p\text{-adic}}(B,\Gamma_1(N)) &\longrightarrow M_{k+2}^{p\text{-adic}}(B,\Gamma_1(N)) \\ \sum_n a_n q^n &\mapsto \sum_n n a_n q^n. \end{aligned} \tag{2.5}$$

This operator does not necessarily preserve overconvergence in general. We do have a special case proven by Coleman:

Theorem 2.1.9 (Theorem 2, [CGJ95])**.** *Let $k\geq 1$ and $f\in M_{1-k}^{oc}(B,\Gamma_1(N))$. Then, $\left(q\frac{d}{dq}\right)^k f\in M_{1+k}^{oc}(B,\Gamma_1(N))$.*

We finish this subsection by explaining that, similarly to classical modular forms over $\mathbb{C}$, we can write q-expansions for overconvergent modular forms. We do so by using a version of the Tate curve, which we also denote by $T(q)$, that is defined over $\mathcal{O}_K$ (cf. Theorem V.3.1 in [Sil94] or Section VII in [DR73]). Indeed, for an overconvergent modular form f of weight k, level Γ and growth condition r as in Definition 2.1.5, we define its q-expansion (at the ∞ cusp) as

$$f(q) := f\left(T(q),\omega_{\text{can}},\iota_{\text{can}},\frac{r}{E_{p-1}\left(T(q),\omega_{\text{can}}\right)}\right)\in B[[q]]. \tag{2.6}$$

One can also define an analog of the above q-expansion of f for every cusp (cf. Section 6.2 in [Gor02]). If every such q-expansion of f lies in $B[[q]]$ instead of $B((q))$, we say that f is holomorphic at the cusps. Moreover, as the Tate curve $T(q)$ is ordinary, $E_{p-1}\left(T(q), \omega_{\mathrm{can}}\right)$ must be invertible in Equation (2.6).

Finally, we generalize the definition of overconvergent modular forms to extensions K of $\mathbb{Q}_p$, instead of just rings of integers $B = \mathcal{O}_K$. We simply take

$$M_k^{\mathrm{oc}}(K, \Gamma; r) := M_k^{\mathrm{oc}}(B, \Gamma; r) \otimes_B K.$$

We endow the space $M_k^{\mathrm{oc}}(K, \Gamma; r)$ with the p-adic topology by specifying that $M_k^{\mathrm{oc}}(B, \Gamma; r)$ will be the unit ball in $M_k^{\mathrm{oc}}(K, \Gamma; r)$. This turns $M_k^{\mathrm{oc}}(K, \Gamma; r)$ into a p-adic Banach space.

2.1.3 Nearly overconvergent modular forms

Nearly overconvergent modular forms are a generalization of overconvergent modular forms, but they still are instances of p-adic modular forms. In recent work of Darmon and Rotger (cf. [DR14]) as well as Urban (cf. [Urb14] and Appendix II of [AI21]), definitions for nearly overconvergent modular forms are given. We will specifically follow Section 3.2 of [Urb14], and direct the reader to read more of [Urb14] for the full details and complete definitions.

Definition 2.1.10 (Nearly overconvergent modular forms)**.** The space of nearly overconvergent modular forms of weight k, level Γ, growth condition $r \in B$ and order of near overconvergence less or equal to $s \in \mathbb{Z}_{\geq 0}$ is given by

$$M_k^{\text{n-oc}}(B, \Gamma; r; s) = H^0\left(\mathcal{X}(\Gamma)_{\leq r}, \omega^{\otimes(k-s)} \otimes \mathrm{Sym}^s(H^1_{\mathrm{dR}})\right),$$

where H^1_{dR} is more precisely defined in Section 2.2 of [Urb14]. We also define the space of nearly overconvergent modular forms of weight k, level Γ, order of near overconvergence less or equal to $s \in \mathbb{Z}_{\geq 0}$ and unspecified growth condition by

$$M_k^{\text{n-oc}}(B, \Gamma; s) = \varinjlim_{\mathrm{ord}_p(r) > 0} M_k^{\text{n-oc}}(B, \Gamma; r; s).$$

In particular, when $s = 0$ we retrieve the usual definition of overconvergent modular forms. The above definition allows us to get the inclusions

$$M_k^{\mathrm{oc}}(B, \Gamma_1(N); r) \subseteq M_k^{\text{n-oc}}(B, \Gamma_1(N); r; s) \subseteq M_k^{p\text{-adic}}(B, \Gamma_1(N))$$

for all r and s. Indeed, this follows from the fact that there is a canonical projection

$$\omega^{\otimes(k-s)} \otimes \mathrm{Sym}^s(H^1_{\mathrm{dR}}) \longrightarrow \omega^{\otimes k}.$$

Definition 2.1.10 has the advantage of being very conceptual and abstract. It also mimics the definition of overconvergent modular forms very closely (see Definition 2.1.6). However, as a consequence, it is hard to use in many practical cases. We therefore give an applied characterization of nearly overconvergent modular forms that makes them easier to grasp.

Recall first the Eisenstein series E_k that give us modular forms (of level 1) and weight k when $k \geq 4$. If we take $k = 2$, the Eisenstein series E_2 is still very interesting, even though it isn't a modular form. It is transcendental over the ring of overconvergent modular forms (cf. [CGJ95]), so

$$M_k^{\mathrm{oc}}(B,\Gamma)(E_2) \cong M_k^{\mathrm{oc}}(B,\Gamma)(X), \tag{2.7}$$

where X is a free variable. It turns out that E_2 plays a useful role for giving an alternative definition of nearly overconvergent modular forms.

Proposition 2.1.11 (Remark 3.2.2 in [Urb14]). *Let $f \in M_k^{n\text{-}oc}(B,\Gamma_1(N);s)$ then there exist overconvergent modular forms $g_0, g_1, ..., g_s$ with $g_i \in M_{k-2i}^{oc}(B,\Gamma_1(N))$ such that*

$$f = g_0 + g_1E_2 + ... + g_sE_2^s. \tag{2.8}$$

By Proposition 2.1.11 and our above explanation (see Equation (2.7)), nearly overconvergent modular forms are polynomials in E_2 with overconvergent modular forms as coefficients, so we can view them as elements of $M_k^{\mathrm{oc}}(B,\Gamma_1(N))(X)$. Recall that we also can see them as being p-adic modular forms. Hence, on top of having a q-expansion in $B[[q]]$, they also have a polynomial q-expansion in $B[[q]][X]$ (of degree less or equal to s, where s is the order of near overconvergence) that comes from Equation (2.8). Consider the operator δ_k taking as input nearly overconvergent modular forms of weight k defined on their polynomial q-expansions as

$$(\delta_k f)(q,X) := q\frac{\mathrm{d}}{\mathrm{d}q}f + kXf(q). \tag{2.9}$$

Then δ_k sends forms of weight k to forms of weight $k+2$. Indeed, this is because the Serre differential operator $q\frac{\mathrm{d}}{\mathrm{d}q}$ increases the weight of its input by 2 as in (2.5). Remember that this operator does not necessarily preserve overconvergence – the best result we have so far regarding this is Theorem 2.1.9. Also, the kXf term on the right hand side of (2.9) denotes kE_2f, which also has weight $k+2$. Define as well the following iterated derivate:

$$\delta_k^s := \delta_{k+2s-2} \circ \delta_{k+2s-4} \circ ... \circ \delta_k.$$

Proposition 2.1.12 (Lemma 3.3.4 in [Urb14]). *Let f be a nearly overconvergent modular form of weight k and order less or equal to s such that $k > 2s$. Then for each $i = 0, ..., s$, there exists a unique overconvergent form h_i of weight $k-2i$ such that*

$$f = \sum_{i=0}^{s} \delta_{k-2i}^i(h_i).$$

Propositions 2.1.11 and 2.1.12 allow us to think about nearly overconvergent forms as having an overconvergent part in them. We define the overconvergent projection π_{oc} of $f = \sum_{i=0}^{s} \delta_{k-2i}^i(h_i)$ by $\pi_{\mathrm{oc}}(f) := h_0$.

Let us now conclude with the following picture of all the different spaces of modular forms that we have seen:

$$
\begin{array}{c}
M^{p\text{-adic}}(\mathbb{Q}_p, \Gamma) \\
\Big| \\
M_k^{p\text{-adic}}(\mathbb{Q}_p, \Gamma) \\
\Big| \\
M_k^{\text{n-oc}}(\mathbb{Q}_p, \Gamma; r; s) \\
\Big| \\
M_k^{\text{oc}}(\mathbb{Q}_p, \Gamma; r) \\
\Big| \\
M_k(\mathbb{Q}_p, \Gamma) = M_k(\mathbb{Q}, \Gamma) \otimes_{\mathbb{Q}} \mathbb{Q}_p \\
\Big| \\
M_k(\mathbb{Q}, \Gamma).
\end{array}
$$

2.2 The U_p operator

2.2.1 U_p acting on classical modular forms

Let $M_k(\mathbb{Q}_p, N, \chi)$ be the space of modular forms of level N, weight k and character χ over $\mathbb{Q}_p$. We could have taken any fixed finite extension K of $\mathbb{Q}_p$ actually, or its ring of integers $B = \mathcal{O}_K$. As in the previous section, when we just say level N here, we mean level $\Gamma_1(N)$. We might drop the $\mathbb{Q}_p$ in the notation of M_k when the base field (or base ring) is obvious. Let $S_k(N, \chi)$ be the subspace of $M_k(N, \chi)$ consisting of cusp forms.

We have the following operators acting on the space of (classical) modular forms: the Hecke operator T_p, the Atkin operator U_p, and the Frobenius operator V,

$$
\begin{aligned}
T_p : \sum_n a_n q^n &\mapsto \sum_n a_{pn} q^n + \chi(p)\, p^{k-1} \sum_n a_n q^{pn}, \\
U_p : \sum_n a_n q^n &\mapsto \sum_n a_{pn} q^n, \\
V : \sum_n a_n q^n &\mapsto \sum_n a_n q^{pn}.
\end{aligned}
\tag{2.10}
$$

The Hecke operator T_p acts on modular forms of level N, for $p \nmid N$. The Atkin operator U_p acts on modular forms of level N, for $p|N$. And lastly, the Frobenius operator V takes modular forms of level N to forms of level pN.

We can think of U_p as collapsing the series or compressing it by removing a big portion of its terms. On the other hand, V can be seen as spacing out a series by adding many zeroes between its terms. In other words, the operator U_p can be viewed as increasing the convergence of a series; whereas V slows it down. This logic will be consistent with the actions of U_p and V on $M_k^{\text{oc}}(N, \chi)$ as we will see next. We also notice that V is a right inverse for U_p and that $VU_p\left(\sum_n a_n q^n\right) = \sum_{p|n} a_n q^n$. In particular, U_p has no left inverse, or else this left inverse would also have to be V. So,

$$U_pV(f) = f, \qquad f^{[p]} := (1 - VU_p)(f) = \sum_{p \nmid n} a_n q^n. \tag{2.11}$$

We call $f^{[p]}$ the p-depletion of f. Note that $1 - VU_p$ is an idempotent operator and we have the formula

$$U_p(V(f) \cdot g) = f \cdot U_p(g), \tag{2.12}$$

which can be proven by looking at q-expansions. In particular, we see that U_p is multiplicative when acting on the product of two forms where one of them is in the image of the *Frobenius* map V.

We now discuss a common tool from linear algebra (particularly useful with infinite dimensional spaces), namely the slope decomposition associated to a linear operator. We are mainly interested in U_p and will therefore introduce the notion of slope α subspace for the U_p operator.

The slope α subspace (whether it is for classical modular forms or for overconvergent modular forms) is the generalized eigenspace of U_p whose eigenvalues have p-adic valuation α. In other words, it's the space of forms f such that there is some integer $r \in \mathbb{N}$ and some $\lambda \in \mathbb{Q}_p$ with valuation $\mathrm{ord}_p(\lambda) = \alpha$ such that $(U_p - \lambda \mathrm{Id})^r(f) = 0$. The following is an alternative definition from [Col97], relying on the concept of Newton polygons.

Definition 2.2.1. A modular form f is said to have slope $\alpha \in \mathbb{Q}$ if there is a polynomial $R(T) \in \mathbb{C}_p[T]$ such that $R(U_p)f = 0$ and such that the Newton polygon of $R(T)$ has only one side and its slope is $-\alpha$. We define the slope α subspace as the space of modular forms of slope α.

In other words, a modular form f has slope α if $R(U_p)(f) = 0$ for some polynomial $R(T)$ of the form $R(T) = 1 + \sum_{i=1}^{\deg(R)} r_i T^i$ such that there exists some $N \leq \deg(R)$ with $\mathrm{val}_p(r_i) \geq -i\alpha$ for all $i < N$ and $\mathrm{val}_p(r_i) = -i\alpha$ for all $i \geq N$.

Example 1. If f is an eigenform of U_p with eigenvalue c of valuation $\mathrm{val}_p(c) = \alpha$, then f is in $M_k(N)^{\text{slope } \alpha}$. Indeed, we can take $R(T) := 1 - T/c$ such that $R(U_p)(f) = 0$ and $R(T)$ has Newton polygon consisting of just one line of slope $-\alpha$.

The action of U_p on $M_k(N)$ gives a U_p-equivariant decomposition of the space of modular forms into subspaces of different slopes,

$$M_k(N) = \bigoplus_{\alpha \in \mathbb{Q}_{\geq 0} \cup \{\infty\}} M_k(N)^{\text{slope } \alpha},$$

where $M_k(N)^{\text{slope } \alpha}$ denotes the subspace of modular forms of slope α. Essentially, $M_k(N)^{\text{slope } \alpha}$ is the generalized eigenspace of U_p whose eigenvalues have p-adic valuation α.

In the case $\alpha = 0$, the modular forms of slope zero are said to be *ordinary* and we denote this space by $M_k^{\text{ord}}(N)$. We now introduce Hida's ordinary projection operator (see Chapter 7.2 of [Hid93] for more details). It is defined as

$$e_{\text{ord}} := \lim U_p^{n!}$$

and projects the entire space $M_k(N)$ onto its subspace of ordinary forms $M_k^{\text{ord}}(N)$.

2.2.2 U_p acting on p-adic modular forms

Let now $M_k^{p\text{-adic}}(B,N)$ denote the space of p-adic modular forms of level N, weight $k \in \mathbb{Z}$ over the ring of integers B of some finite extension of $\mathbb{Q}_p$, as defined in Section 2.1.2 (see also [Ser73]). As in the above, we might drop the B in the notation of $M_k^{p\text{-adic}}$ when the base ring is obvious. This is an infinite dimensional space.

Since p-adic modular forms also have q-expansions just like classical modular forms, we can let the operators T_p, U_p and V act on $M_k^{\text{oc}}(N)$, using the same definitions as in (2.10). There also exist definitions for T_p, U_p and V where we view modular forms as functions of elliptic curves with extra structure (test objects) rather than just letting T_p, U_p and V act on q-expansions. These definitions for T_p, U_p and V are good for certain conceptual aspects. However, we will not see them here as we are mainly concerned with computational applications.

We will see that the U_p operator, when acting on $M_k^{p\text{-adic}}(B,N)$, isn't very interesting and we will then restrict the domain of U_p to the space of overconvergent modular forms $M_k^{\text{oc}}(B,N)$.

Indeed, we first notice that given any $f \in M_k^{p\text{-adic}}(B,N) = M_k^{p\text{-adic}}(B,N;1)$, we can define a new modular form

$$f_0 := f^{[p]} := (1 - VU_p)(f)$$

that lies in $\ker(U_p)$. Moreover, given any $f_0 \in \ker(U_p)$ (which we just saw how to create out of any $f \in M_k^{p\text{-adic}}(B,N)$) and any $\lambda \in B$ with $\text{ord}_p(\lambda) > 0$, we can define a new p-adic modular form

$$f_\lambda := f_0 + \lambda V(f_0) + \lambda^2 V^2(f_0) + \lambda^3 V^3(f_0) + \ldots$$

This p-adic modular form is well defined as the above series converges (because $\text{ord}_p(\lambda) > 0$). We thus get an eigenform f_λ for U_p of eigenvalue λ.

Therefore, we can conclude that $\ker(U_p) \cong \ker(U_p - \lambda)$ for all λ in the maximal ideal of B (i.e. such that $\text{ord}_p(\lambda) > 0$). Note as well that this construction ensures that f, f_0, f_λ have the same q-expansion outside of (p), i.e.

$$a_n(f) = a_n(f_0) = a_n(f_\lambda)$$

whenever $p \nmid n$.

2.2.3 U_p acting on overconvergent modular forms

We now restrict our attention to the space $M_k^{\text{oc}}(\mathbb{Q}_p, N)$ of overconvergent modular forms of level N, weight k and character χ over $\mathbb{Q}_p$, as defined in Section 2.1.2 (see also [Kat73]). As in the above, we might drop the $\mathbb{Q}_p$ in the notation of M_k^{oc} when the base field is obvious. This is an infinite dimensional space, but it is not as big as $M_k^{p\text{-adic}}(\mathbb{Q}_p, N)$.

As we have seen above for p-adic modular forms, overconvergent modular forms also have q-expansions, and we can let T_p, U_p and V act on $M_k^{\text{oc}}(N)$, using the same definitions as in (2.10). We will see here the action of U_p on $M_k^{\text{oc}}(N)$ will be much more interesting than its action on $M_k^{p\text{-adic}}(N)$.

First of all, note that U_p doesn't necessarily preserve the growth conditions of an overconvergent modular form. However, if we restrict our attention to the case $0 < \mathrm{ord}_p(r) < \frac{1}{p+1}$, then we have an inclusion

$$p \cdot U_p : M_k^{\mathrm{oc}}(B, N; r) \hookrightarrow M_k^{\mathrm{oc}}(B, N; r^p),$$
$$f \mapsto p \cdot U_p(f),$$

as in Lemma 3.11.4 of [Kat73]. So $U_p\left(M_k^{\mathrm{oc}}(B, N; r)\right) \subseteq \frac{1}{p} M_k^{\mathrm{oc}}(B, N; r^p)$. Combining this with the fact that $M_k^{\mathrm{oc}}(B, N; r^p) \subseteq M_k^{\mathrm{oc}}(B, N; r)$ via the map in (2.4), we can view the Atkin operator U_p as an endomorphism of $M_k^{\mathrm{oc}}(K, N; r)$ when $0 < \mathrm{ord}_p(r) < \frac{p}{p+1}$. Of course, U_p can always be seen as an endomorphism of $M_k^{p\text{-adic}}(B, N; 1)$.

Note that we had to tensor $M_k^{\mathrm{oc}}(B, N; r)$ with K to get that U_p can be viewed as an endomorphism of $M_k^{\mathrm{oc}}(K, N; r)$. In addition, in this case, U_p is a completely continuous endomorphism. Hence, we can apply p-adic spectral theory, as in [Ser62]. We therefore obtain that the Atkin operator U_p will induce a decomposition (as in Section 2 of [Wan98]), for all $\alpha \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$, on the space of overconvergent modular forms

$$M_k^{\mathrm{oc}}(K, N; r) = M_k^{\mathrm{oc}}(K, N; r)^{\text{slope } \alpha} \oplus X_\alpha, \tag{2.13}$$

where $M_k^{\mathrm{oc}}(K, N; r)^{\text{slope } \alpha}$ is the finite dimensional space of overconvergent modular forms in $M_k^{\mathrm{oc}}(K, N; r)$ of slope α. If we further assume an infinite slope version of the spectral expansion conjecture (cf. [GM95]), we would obtain

$$M_k^{\mathrm{oc}}(K, N; r) = \widehat{\bigoplus_{\alpha \in \mathbb{Q}_{\geq 0} \cup \{\infty\}}} M_k^{\mathrm{oc}}(K, N; r)^{\text{slope } \alpha}, \tag{2.14}$$

for $\mathrm{ord}_p(r) \in \left(\frac{1}{p+1}, \frac{p}{p+1}\right)$, where $\widehat{\oplus}$ denotes the completed direct sum. Note that partial results towards the spectral expansion conjecture have been obtained in [Loe07] when $p = 2$. Similarly to classical modular forms, the overconvergent modular forms of slope zero are said to be ordinary and we denote this space by $M_k^{\mathrm{oc,ord}}(N)$. Actually, Coleman's classicality theorem (cf. [Col95]) states that any ordinary overconvergent modular form of weight $k \geq 2$ can be seen as a classical modular form of weight k on $\Gamma_1(N)$. Therefore, when $k \geq 2$, we can simply denote the ordinary overconvergent modular by $M_k^{\mathrm{ord}}(N)$ instead of $M_k^{\mathrm{oc,ord}}(N)$.

Hida's ordinary projection operator $e_{\mathrm{ord}} := \lim U_p^{n!}$ also acts on p-adic modular forms and in particular on overconvergent modular forms. Hida's operator projects the entire space $M_k^{\mathrm{oc}}(N)$ onto its subspace of ordinary forms $M_k^{\mathrm{oc,ord}}(N)$.

Following Darmon and Rotger in [DR14] (see Lemma 2.17), we notice that $a_n(g^{[p]} \times (Vh)) = 0$ whenever $p|n$. This gives the following lemma, which is also a direct consequence of equations (2.11) and (2.12).

Lemma 2.2.2. *If g and h are p-adic modular forms, then $g^{[p]} \times (Vh)$ is in the kernel of the U_p operator, and in particular $e_{ord}(g^{[p]} \times (Vh)) = 0$.*

2.2.4 U_p acting on nearly overconvergent modular forms

Consider now the space of nearly overconvergent modular forms. One can still define the ordinary projection operator as $e_{\mathrm{ord}} := \lim U_{p^{n!}}$, since this operator is actually defined for any p-adic modular form in general. It turns out that the ordinary projection of a nearly overconvergent modular form only depends on its overconvergent part.

Theorem 2.2.3 (Lemma 2.7 in [DR14])**.** *Let F be a nearly overconvergent modular form, then*

$$e_{ord}(\phi) = e_{ord}\pi_{oc}(\phi).$$

Thus, taking ordinary projections of *nearly overconvergent* modular forms reduces to taking ordinary projections of *overconvergent* modular forms.

As explained in Section 3.3.6 of [Urb14] (see also Appendix II of [AI21] for Urban's erratum to [Urb14]), the Atkin operator U_p is also completely continuous when viewed as an endomorphism of $M_k^{\text{n-oc}}(K, N; r; s)$, for $0 < \text{ord}_p(r) < \frac{1}{p+1}$. Hence, we obtain a decomposition of $M_k^{\text{n-oc}}(K, N; r; s)$ similar to that of Equation (2.13). This means that we may also similarly speak of slope α projections $e_{\text{slope }\alpha}(\psi)$ for nearly overconvergent modular forms ψ.

For our work, we are interested in computing ordinary projections of overconvergent modular forms (Section 3.1.2). But we will also consider computing ordinary projections of *nearly* overconvergent modular forms (Section 3.1.3).

One could also be interested in taking projection over different slope spaces. What we will focus on in Section 3.2 however, is how to take a projection over the space generated by a particular element f_α in the space of forms of slope α, instead of projecting over the entire space of slope α.

2.3 The Poincaré pairing

To a cuspidal modular form $\phi = \sum_n a_n(\phi)q^n$ of weight two and level N, one can associate a differential $\omega_\phi \in H^1_{\text{dR}}(X_1(N))$ given by

$$\omega_\phi = \phi(q)\frac{\mathrm{d}q}{q} = \sum_n a_n(\phi)q^n\frac{\mathrm{d}q}{q}. \tag{2.15}$$

In general, given a modular form ϕ of weight $r+2$ and level $\Gamma_1(N)$, one can associate to it a differential $\omega_\phi \in \text{Fil}^{r+1} H^{r+1}_{\text{dR}}(\mathcal{E}^r/\mathbb{C}_p)$, where $\mathcal{E}$ is the universal generalized elliptic curve fibered over $X_1(N)$, and $\mathcal{E}^r$ is the Kuga-Sato variety as in [Sch90]. In such a case, writing down a closed formula for ω_ϕ is possible, but more tricky and less simple that the formula $\omega_\phi = \sum_n a_n(\phi)q^n\frac{\mathrm{d}q}{q}$ obtained in the weight 2 case. Indeed, we would get

$$\omega_\phi(q) = \sum_n a_n(\phi)q^n\omega_{\text{can}}^r\left(\frac{\mathrm{d}q}{q}\right),$$

on the Tate curve $T(q)$ with canonical differential ω_{can}. See Section 2.2 of [DR14] for more details.

The ϕ-isotypic component of $H^{r+1}_{\text{dR}}(\mathcal{E}^r/\mathbb{C}_p)$, denoted $H^{r+1}_{\text{dR}}(\mathcal{E}^r/\mathbb{C}_p)_\phi$ is two dimensional. Assume now that ϕ is ordinary at p. This implies the existence of a one dimensional subspace (the *unit root subspace*) on which the Frobenius endomorphism acts as multiplication by a p-adic unit. We can then pick a unique element $\eta_\phi^{\text{u-r}}$ in this unit root subspace to extend $\{\omega_\phi\}$ to a basis $\{\omega_\phi, \eta_\phi^{\text{u-r}}\}$ such that $\langle \omega_\phi, \eta_\phi^{\text{u-r}} \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the alternating Poincaré duality pairing on $H_{\text{dR}}^{r+1}(\mathcal{E}^r/\mathbb{C}_p)$. The basis $\{\omega_\phi, \eta_\phi^{\text{u-r}}\}$ will play an extremely important role in Section 4. We will now provide some more information about the Poincaré pairing. The reader can consult Chapter 5 of [Col95] for more on this pairing. We will however follow the notation and conventions used in Section 2.1 of [DR14].

Let $\mathcal{X}$ be an arbitrary smooth proper curve over $\text{Spec}(\mathbb{Z}_p)$ and let X be its generic fibre. For a complete subfield K of $\mathbb{C}_p$, we write X_K for the base change of X to K. The authors in *loc. cit.* construct open annuli $\mathcal{V}_1, ..., \mathcal{V}_s$ and a region

$$\mathcal{W}_\varepsilon = \mathcal{A} \cup \bigcup_{i=1}^{s} \mathcal{V}_j,$$

such that the de Rham cohomology $H_{\text{dR}}^1(X_K)$ can be identified with the space of classes of rigid analytic forms on $\mathcal{W}_\varepsilon$ over K with vanishing annular residues.

Let us write $\Omega^1(\mathcal{W}_\varepsilon)$ for the set of rigid differentials on $\mathcal{W}_\varepsilon$. Given two cohomology classes ξ_1, ξ_2 in $H_{\text{dR}}^1(X_K)$, let $\omega_1, \omega_2 \in \Omega^1(\mathcal{W}_\varepsilon)$ be respective representatives for them. Let also $F_{\omega_1}^{(j)}$ be a local analytic primitive of ω_1 on the annulus $\mathcal{V}_j$. The Poincaré pairing is then given by

$$\langle \xi_1, \xi_2 \rangle = \sum_{j=1}^{s} \text{Res}_{\mathcal{V}_j} \left(F_{\omega_1}^{(j)} \cdot \omega_2 \right), \tag{2.16}$$

where $\text{Res}_{\mathcal{V}_j}$ is the p-adic annular residue (cf. Chapter 7 of [Col94] and Lemma 2.1 of [Col89]).

In practice, the difficulty in computing a Poincaré pairing lies in finding appropriate expansions for the terms appearing in Equation (2.16). We will discuss this in Chapter 6, as well as a known method for computing Poincaré pairings, which only works in a very limited number of cases. We will also explain how the work done in this thesis allows us to compute more general instances of Poincaré pairings.

Chapter 3

Explicit algorithmic methods

This chapter is one of the two main component of this thesis. In it we will introduce new algorithmic methods that build upon the current known methods. We first start by recalling the work of Lauder, in [Lau14], on ordinary projection of overconvergent modular forms. We then explain how to generalize it to nearly overconvergent modular forms. Finally, we explain how one can actually compute general projections (not just ordinary projections). This will enable us to study more general theoretical objects in Section 4.2.

3.1 Ordinary projections

3.1.1 Computing the Katz Basis and the U_p operator

In this thesis, we are interested in computational applications. We would therefore like to be able to write down q-expansions for our overconvergent modular forms. However, as these q-expansions are simply power series in p-adic terms, we will have to approximate our overconvergent modular forms as truncated power series (i.e. polynomials) modulo p^n by viewing them in $\mathbb{Z}[[q]]/(q^h, p^m)$ for some $m \in \mathbb{N}$ and some $h \in \mathbb{N}$ depending on N, k and χ. Once we know what level of precision we want to obtain after our calculations, we can decide what level of precision we need to start with, as we know how much precision is lost through the algorithms that we use. An alternative – more ad hoc – way to measure the precision of our outputs (p-adic numbers) is to run our algorithm multiple times, to different precisions, and see by what power of p they differ.

We will explain how to write the q-expansion of an overconvergent modular form $H \in M_k^{\mathrm{oc}}(\mathbb{Z}_p, N, \chi; \frac{p}{p+1})$ and write down a matrix representing U_p. We follow [Lau14] to do so. Assume χ is trivial for simplicity. Our output will have level of precision $\mathbb{Z}[[q]]/(p^m, q^{h(m,p)})$ for any desired $m \in \mathbb{N}$ and where $h(m,p)$ is defined in the following. If we want our output to lie in $\mathbb{Z}[[q]]/(p^m, q^{h(m,p)})$, we need to start with a greater level of precision, because our method might cause a certain loss of precision. In [Lau14], we see that we need to take $n := \left\lfloor \frac{(p+1)m}{p-1} \right\rfloor$ and let $h(m,p) := h'p$, where h' is the *Sturm bound* of $M_{k+n(p-1)}(N)$.

Let $d_i := \dim_{\mathbb{Z}_p} M_{k+i(p-1)}(\mathbb{Z}_p, N)$ and $m_i := d_i - d_{i-1} = \dim_{\mathbb{Z}_p} A_{k+i(p-1)}(\mathbb{Z}_p, N)$. Let also $d := d_n$. Pick a row-reduced basis $\{b_{i,s} : s = 1, ..., m_i\}$ for each $A_{k+i(p-1)}(\mathbb{Z}_p, N)$ for $i = 0, ..., n$. Compute

$$e_{i,s} := \frac{p^{\left\lfloor \frac{i}{p+1} \right\rfloor} \cdot b_{i,s}}{E_{p-1}^i} \mod (q^{h'p}, p^{m'}),$$

where $m' := m + \left\lceil \frac{n}{p+1} \right\rceil$. Let

$$\mathrm{Kb} := \{e_{i,s}\}_{i,s} = \left\{ \frac{p^{\left\lfloor \frac{i}{p+1} \right\rfloor} \cdot b}{E_{p-1}^i} \mod (q^{h'p}, p^{m'}) : i = 1, ..., \left\lfloor \frac{(p+1)m}{p-1} \right\rfloor ; s = 1, ..., m_i \right\}.$$

We call Kb the *Katz basis*, it has d_n elements. To simplify notation, we shall write $\mathrm{Kb} = \{v_1, ..., v_{d_n}\}$.

Any overconvergent modular form of growth condition $r := \frac{1}{p+1}$, when reduced modulo $(q^{h'p}, p^{m'})$, can be expressed as a linear combination of elements in Kb.

Now, we apply the Atkin operator U_p to the Katz basis to obtain

$$t_{i,s} := U_p(e_{i,s}) \mod (q^{h'}, p^{m'}),$$

and write $S := \{t_{i,s}\}$. Let E and T be the $d \times h'$ matrices formed by taking the elements of Kb and S respectively and looking at the first h' terms in their q-expansions. Using linear algebra, compute the $d \times d$ matrix A' such that $T = A'E$. Then, $A := A' \mod p^m$ is the representation of the operator U_p in the Katz basis. We write $A = [U_p]_{\mathrm{Kb}}$.

The advantage of this approach is that we only need to compute U_p once on the Katz basis and then we will be able to apply the Atkin operator as many times as we wish without having to actually use its original definition. Given an overconvergent modular form f of growth condition $\frac{p}{p+1}$, we can express it as a sum

$$f = \sum_i \alpha_i v_i \mod (q^{h'p}, p^{m'}). \tag{3.1}$$

Write $[f]_{\mathrm{Kb}} := (\alpha_1, ..., \alpha_d)$ and compute $A[f]_{\mathrm{Kb}}$. Letting γ_i denote the entries of $A[f]_{\mathrm{Kb}}$, we find

$$U_p(f) = \sum_i \gamma_i v_i \mod (q^{h'p}, p^m).$$

Thus,

$$[U_p(f)]_{\mathrm{Kb}} = A[f]_{\mathrm{Kb}}. \tag{3.2}$$

The proof of correctness of Equation (3.2) is quite subtle. Indeed, we are representing the infinite matrix U_p by a $d \times d$ truncation A of it. In general, there would be no reason for this to hold. However, there are two results that allow us to justify this step. First of all, Wan shows in [Wan98] that the entries of the matrix U_p decay as we go down along the columns. We thus obtain, in our setting, that the bottom part of U_p (below the first d entries) vanishes modulo p^m. This result, on its own, is not enough to justify Equation (3.2), as we still need to deal with the right portion of the matrix U_p, and show that its entries also decay (and vanish modulo p^m) as we go along the rows. In other words, we are interested in the question mark "?" appearing in the equation

$$U_p = \left[\begin{array}{c|c} A & ? \\ \hline 0 & 0 \end{array}\right] \mod p^m.$$

We cannot actually show that the entries of U_p decay as we go along the rows, which would have been enough to justify Equation (3.2). Instead, we turn our attention the vector $[f]_{\mathrm{Kb}}$. In general, when f is $1/(p+1)$-overconvergent, we cannot do much. However, as is explained in Section 2.2.2 of [Lau14] and the last paragraph of Section 3.2.1 in [Lau11], when f is $p/(p+1)$-overconvergent, the coordinates of f, when represented as an infinite vector in the infinite Katz Basis, vanish modulo p^m, except for the first d entries. We thus get, modulo p^m,

$$\left[\begin{array}{c|c} A & ? \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} [f]_{\mathrm{Kb}} \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} A[f]_{\mathrm{Kb}} \\ \hline 0 \end{array}\right],$$

which gives Equation (3.2).

Remark 6. Note that we let the overconvergent modular form f in Equation (3.1) have growth rate $\frac{p}{p+1}$ instead of just $\frac{1}{p+1}$. Although we can write a $\frac{1}{p+1}$-overconvergent modular form ϕ in the Katz basis, and $A = [U_p]_{\mathrm{Kb}}$ in the same basis, we cannot directly apply A to $[\phi]_{\mathrm{Kb}}$, as we just explained in previous paragraph. Indeed, the coefficients in the expansion of $[\phi]_{\mathrm{Kb}}$ will not decay fast enough (p-adically) for our calculations to be accurate. This issue is entirely avoided when ϕ is $\frac{p}{p+1}$-overconvergent. Thus, when dealing with a $\frac{1}{p+1}$-overconvergent form ϕ, we have to compute $U_p(\phi)$ directly (without using the matrix representation A of U_p) to obtain a $\frac{p}{p+1}$-overconvergent form, thus improving its overconvergence and decay properties. After that, we may apply A to $[U_p(\phi)]_{\mathrm{Kb}}$.

3.1.2 Ordinary projections of overconvergent modular forms

As we just saw, we can write a matrix A representing the operator U_p in the Katz basis $\mathrm{Kb} = \{e_{i,s}\}_{i,s}$ modulo p^m and q^h for some appropriately chosen $m, h \in \mathbb{N}$.

To compute the ordinary projections of overconvergent modular forms, we recall the definition $e_{\mathrm{ord}} := \lim U_{p^{n!}}$. So, having represented U_p as a matrix A, we need to pick a big enough $R \in \mathbb{N}$ such that A^R represents e_{ord} to our desired level of precision. We see from Algorithm 2.1 in [Lau14] that we can take $R := (p^\kappa - 1)p^m$ and κ is a positive integer such that all the unit roots of the reverse characteristic polynomial of A lie in some extension of $\mathbb{Z}_p$ with residue class field of degree κ over $\mathbb{F}_p$.

So, given an overconvergent modular form f of growth condition $\frac{p}{p+1}$, written as $\sum_i \alpha_i v_i$ modulo $(q^{h'}, p^n)$, we compute $\gamma := A^R[f]_{\mathrm{Kb}}$ and let γ_i denote the entries of γ. Finally, we obtain

$$e_{\mathrm{ord}}(f) = \sum_i \gamma_i v_i \mod (q^{h'}, p^n).$$

3.1.3 Ordinary projections of nearly overconvergent modular forms

For simplicity, let d denote the operator $q\frac{\mathrm{d}}{\mathrm{d}q}$. So the δ_k operator from (2.9) becomes $\delta_k = \mathrm{d} + kX$. Let g, h be two classical modular forms of weights ℓ, m respectively, and let $H := \mathrm{d}^{-(1+t)}(g^{[p]}) \times h$, for some integer t with $0 \le t \le \min\{\ell, m\} - 2$. We wish to compute

$$\mathcal{X} := e_{\mathrm{ord}}(H) = e_{\mathrm{ord}}\left(\mathrm{d}^{-(1+t)}(g^{[p]}) \times h\right).$$

The modular form $\mathrm{d}^{-(1+t)}(g^{[p]})$ has weight $\ell - 2(1+t)$, hence $\mathcal{X}$ has weight $\ell + m - 2t - 2$. The condition $0 \le t \le \min\{\ell, m\} - 2$ ensures that $\mathcal{X}, g$ and h are balanced, i.e. the largest weight is strictly smaller than the sum of the other two. In other words, three modular forms are balanced if their weights are the lengths of the sides of a triangle of non-zero area.

If we had that $t = \ell - 2$, the form $H := \mathrm{d}^{-(1+t)}(g^{[p]}) \times h$ would have been overconvergent. This is because Theorem 2.1.9 still applies for negative powers of d, after depleting the modular form to avoid dividing by p, i.e. we have a map

$$\begin{aligned} M^{\mathrm{oc}}_{1+a}(B, \Gamma_1(N)) &\longrightarrow M^{\mathrm{oc}}_{1-a}(B, \Gamma_1(N)) \\ g &\mapsto \mathrm{d}^{-a} g^{[p]} = \sum_{p \nmid n} \frac{a_n(g)}{n^a} q^n, \end{aligned} \tag{3.3}$$

for all $a \ge 1$. But, in our case, H is not necessarily overconvergent and we cannot directly use the methods introduced in [Lau14] to compute the ordinary projection $e_{\mathrm{ord}}(H)$. However, H is nearly overconvergent (Proposition 2.9 in [DR14]) and Theorem 2.2.3 tells us that

$$e_{\mathrm{ord}}(H) = e_{\mathrm{ord}}\left(\pi_{\mathrm{oc}}(H)\right) = e_{\mathrm{ord}}\left(\pi_{\mathrm{oc}}\left(\mathrm{d}^{-(1+t)}(g^{[p]}) \times h\right)\right),$$

where π_{oc} is the overconvergent projection operator. Since $\pi_{\mathrm{oc}}(H)$ is overconvergent, by definition, we can follow the methods described in [Lau14] to compute its ordinary projection, thus obtaining $e_{\mathrm{ord}}\left(\pi_{\mathrm{oc}}(H)\right) = e_{\mathrm{ord}}(H)$. We therefore turn our attention to computing $\pi_{\mathrm{oc}}(H)$. Note that we are not actually interested in taking the overconvergent projection of any nearly overconvergent modular form; we are specifically computing $\pi_{\mathrm{oc}}\left(\mathrm{d}^{-(1+t)}(g^{[p]}) \times h\right)$. We therefore use a trick (see Theorem 3.1.1) that specifically applies to our setting, as was suggested to us by David Loeffler.

Set $G := \mathrm{d}^{1-\ell} g^{[p]}$, it is an overconvergent modular form of weight $2 - \ell$, as in Equation (3.3). Let $n = \ell - 2 - t \ge 0$ so that $\mathrm{d}^{-1-t} g^{[p]} = \mathrm{d}^n G$ and $\pi_{\mathrm{oc}}\left(\mathrm{d}^{-1-t}(g^{[p]}) \times h\right) = \pi_{\mathrm{oc}}((\mathrm{d}^n G) \times h)$. Consider the Rankin-Cohen bracket [Coh75, Zag94]

$$[G, h]_n = \sum_{a,b \ge 0, a+b=n} (-1)^b \binom{(2-\ell)+n-1}{b} \binom{m+n-1}{a} \mathrm{d}^a(G)\mathrm{d}^b(h). \tag{3.4}$$

Note that the individual terms in this sum are all p-adic modular forms of weight $\ell + m - 2t - 2$ that are not necessarily overconvergent. However, the entire sum $[G, h]_n$ is overconvergent (see Theorem 3.1.1 below). It turns out that the Rankin-Cohen bracket is closely related to the overconvergent projection operator.

Theorem 3.1.1. *Let ϕ_1, ϕ_2 be overconvergent modular forms of weights κ_1 and κ_2 respectively, then, for all $s \geq 0$,*

$$[\phi_1, \phi_2]_s = \binom{\kappa_1 + \kappa_2 + 2s - 2}{s} \pi_{oc}((\mathrm{d}^s \phi_1) \times \phi_2).$$

This follows from Section 4.4 of [LSZ20] (see also Theorem 1 in [Lan08]). We thus obtain the following Corollary.

Corollary 3.1.2. *We can relate $[G, h]_n$ and $\pi_{oc}((\mathrm{d}^n G) \times h)$ as follows*

$$[G, h]_n = \binom{-\ell + m + 2n}{n} \pi_{oc}((\mathrm{d}^n G) \times h).$$

Thus, we can simply compute $[G, h]_n$ using equation (3.4) to obtain $\pi_{\mathrm{oc}}(H)$.

Remark 7. Note that we had to pass through G instead of using g directly as we cannot have the subscript s of the Rankin-Cohen bracket $[\cdot, \cdot]_s$ be negative. Moreover, since the modular forms $\mathcal{X}, g, h$ are balanced, $\binom{-\ell+m+2n}{n}$ cannot be zero.

3.2 Eigenspace σ projections

In the previous sections, we have seen how to compute ordinary projections, i.e. projections over the space of overconvergent modular forms of slope zero. We now consider taking projections over the space of overconvergent modular forms of slope α, for any $\alpha \in \mathbb{Q}_{\geq 0}$, as defined in Section 2.2, via the slope decomposition in Equation (2.13).

We start by making this notion of projection clear. Recall the U_p equivariant decomposition of $M_k^{\mathrm{oc}}(N)$ described in (2.13). For all $\alpha \in \mathbb{Q} \cup \{\infty\}$, it allows us to express any form H as a sum $H = F_\alpha + F$, where $F_\alpha \in M_k^{\mathrm{oc}}(N)^{\text{slope } \alpha}$ and $F \in X_\alpha$. We then call F_α the projection of H onto the space of slope α, or the slope α projection of H. Consider now the eigenspace associated to a single eigenvalue σ such that $\mathrm{val}_p(\sigma) = \alpha$. We will explain how to project modular forms onto such an eigenspace. This method has been used in [DL21] and is based on an insight of David Loeffler (see the last paragraph of Section 6.3 of [LSZ20]). We call such a projection the eigenspace σ projection. This can be seen as a special case of the slope α projection, as these two notions would agree in the case where U_p only has one eigenvalue σ of valuation α.

In order to compute a eigenspace σ projection, we will use the following algebra trick rather than doing it directly. To do so, we will use the Smith normal form (cf. [Smi61]).

Theorem 3.2.1 (Smith's normal form)**.** *Fix a principal ideal domain R and let $M \in M_n(R)$. Using elementary operations, one can transform M to a matrix D of the form*

$$D = diag\,(a_1, ..., a_s)\,,$$

such that $a_1, ..., a_s$ are the invariant factors of M. We also have $a_1 | a_2 | ... | a_s$. In particular, there are invertible matrices P, Q such that $QMP = D$.

Remark 8. When we say elementary operations in Theorem 3.2.1, we mean the following operations:

(i) Exchanging 2 rows, or 2 columns.

(ii) Adding an $\mathbb{F}[x]$-multiple of a row to another row, and the same with columns.

(iii) Multiplying a row, or a column, by a unit of $\mathbb{F}[x]$ (i.e. a non-zero scalar in $\mathbb{F}$).

There are efficient computational ways to find the expression $QMP = D$ described in Theorem 3.2.1, and we can use MAGMA ([BCP97]) to do so in practice.

Recall that in Section 3.1.1 we found a matrix A representing the Atkin operator U_p acting on the Katz basis of $M_k^{oc}(\mathbb{Z}_p, N, \chi; \frac{1}{p+1})$. Let σ be a eigenvalue for U_p and let $M = M_\sigma := A - \sigma \mathrm{Id}$. We can put M_σ in *Smith normal form* D, where

$$D = \mathrm{diag}\left(a_1(\sigma), ..., a_{s-1}(\sigma), a_s(\sigma)\right), \tag{3.5}$$

such that $a_1(\sigma)|a_2(\sigma)|...|a_s(\sigma)$. Let $P = P_\sigma$ and $Q = Q_\sigma$ be the matrices such that $QM_\sigma P = D$. We now remark that $a_s(\sigma)$ should be zero, as σ is an eigenvalue for U_p. However, recall that A is only an approximation for U_p. More precisely (see Section 3.1.1), $A \in M_{d_n \times d_n}(\mathbb{Z}/p^m\mathbb{Z})$ is equal to U_p modulo p^m. And so, $a_s(\sigma)$ will only be zero in $\mathbb{Z}/p^m\mathbb{Z}$.

Moreover, the case $a_{s-1}(\sigma) = 0$ happens precisely when σ has multiplicity (as an eigenvalue of U_p) more than one. We will initially exclude this case for simplicity, but we will address it later in Section 3.2.2. Indeed, in the case where σ has multiplicity more than one, the eigenspace associated to σ also contains another eigenform (other than the one we are projecting on) and the method we are presenting in Section 3.2.1 will not work.

Assume henceforth that we are dealing with an eigenvalue σ of multiplicity one. In particular, the σ-eigenspace is one-dimensional. This assumption will be crucial in Section 3.2.1.

3.2.1 The projector to f_σ

For the reminder of this section, we will assume the spectral expansion formula given by Equation (2.14). The spectral expansion conjecture [GM95] is widely believed to be true and has been proven in the case where $p = 2$, $N = 1$ and $5/12 < r < 7/12$ (cf. [Loe07]). Our algorithm for eigenspace σ projections will thus work under the assumption that this conjecture holds.

Let f_σ be an eigenform lying in the one-dimensional σ-eigenspace. Let $\pi := \pi_\sigma$ denote the last row of $Q \in M_{d_n \times d_n}(\mathbb{Z}/p^m\mathbb{Z})$, i.e. $\pi_i = Q_{i,d_n}$ for $i = 1, ..., d_n$. We call π the *projector* to f_σ. The reason for this will become clear in the following. Recall from Equation (3.5) that $QMP = D = \mathrm{diag}\left(a_1(\sigma), ..., a_{s-1}(\sigma), 0\right)$ and $\sum_{k,l} Q_{ik} M_{kl} P_{lj} = D_{ij}$, where $M := A - \sigma \mathrm{Id}$. We will show that π is orthogonal to all modular forms of eigenvalue not σ.

Proposition 3.2.2. *The projector π_σ is orthogonal to all $p/(p+1)$-overconvergent modular forms (written in the Katz basis) not in the σ-eigenspace.*

Proof. As we are working with $p/(p+1)$-overconvergent modular forms, we will be able to represent the action of U_p on them by the matrix A given in Section 3.1.1. See, in particular, the final paragraph of that section for more details on representing U_p by A.

We start with the simplest case. Let f_s be an eigenform of U_p with eigenvalue s, such that $s \neq \sigma$. Then, $M[f_s]_{\mathrm{Kb}} = (A - \sigma \mathrm{Id})[f_s]_{\mathrm{Kb}} = (s - \sigma)[f_s]_{\mathrm{Kb}}$. Hence,

$$Q(s-\sigma)[f_s]_{\mathrm{Kb}} = QM[f_s]_{\mathrm{Kb}} = DP^{-1}[f_s]_{\mathrm{Kb}}. \tag{3.6}$$

Since π is the last row of Q and the last row of D is completely zero, Equation (3.6) gives

$$(s-\sigma)\pi[f_s]_{\mathrm{Kb}} = \pi(s-\sigma)[f_s]_{\mathrm{Kb}} = 0. \tag{3.7}$$

As $s \neq \sigma$, we must have $\pi[f_s]_{\mathrm{Kb}} = 0$, up to a certain level of precision, as is explained in Remark 9. This shows that any eigenform of U_p, with eigenvalue of different norm than the norm of σ, is orthogonal to π.

Let now F_s be a generalized eigenform for the eigenvalue s, again with $s \neq \sigma$. There exists some minimal integer $r \in \mathbb{N}$ such that $(A - s\mathrm{Id})^r[F_s]_{\mathrm{Kb}} = 0$. Let $M_s := A - s\mathrm{Id}$, so that $M_s^r[F_s]_{\mathrm{Kb}} = 0$, and write

$$\begin{aligned}(M - M_s)^r\,[F_s]_{\mathrm{Kb}} &= \sum_{i=0}^{r-1}\binom{r}{i}(-1)^i M^{r-i}M_s^i[F_s]_{\mathrm{Kb}} + (-1)^r M_s^r[F_s]_{\mathrm{Kb}} \\ &= M\sum_{i=0}^{r-1}\binom{r}{i}(-1)^i M^{r-1-i}M_s^i[F_s]_{\mathrm{Kb}}.\end{aligned}$$

Therefore, $(M - M_s)^r\,[F_s]_{\mathrm{Kb}} = MC[F_s]_{\mathrm{Kb}}$, where $C := \sum_{i=0}^{r-1}\binom{r}{i}(-1)^i M^{r-1-i}M_s^i$. Now, $(M - M_s)^r = (s-\sigma)^r\mathrm{Id}$, hence

$$(s-\sigma)^r \cdot Q[F_s]_{\mathrm{Kb}} = Q\,(M - M_s)^r\,[F_s]_{\mathrm{Kb}} = QMC[F_s]_{\mathrm{Kb}} = DP^{-1}C[F_s]_{\mathrm{Kb}}. \tag{3.8}$$

And as above, Equation (3.8) gives

$$(s-\sigma)^r\pi[F_s]_{\mathrm{Kb}} = 0. \tag{3.9}$$

Finally, since $s \neq \sigma$, we have $\pi[F_s]_{\mathrm{Kb}} = 0$, up to a certain level of precision (see Remark 9). That is, π must be orthogonal to all overconvergent modular forms not in the σ-eigenspace. □

Remark 9. It is crucial in Equations (3.7) and (3.9) that we are working over $\mathbb{Z}_p$ in order to conclude that $\pi[f_s]_{\mathrm{Kb}}$ and $\pi[F_s]_{\mathrm{Kb}}$ are zero. However, in practice, we are working over $\mathbb{Z}/p^m\mathbb{Z}$ for some $m \in \mathbb{Z}$. So Equation (3.9) actually becomes $p^m|(s-\sigma)^r\pi[f_s]_{\mathrm{Kb}}$, which does not necessarily imply that $p^m|\pi[f_s]_{\mathrm{Kb}}$. Therefore, there is a loss of precision of $r \cdot \mathrm{val}_p(s-\sigma)$. This loss of precision can be bounded above by looking at the largest non-zero entry of D, since $\mathrm{val}_p(s-\sigma) \leq \max_i \mathrm{val}_p(D_{i,i})$. To see this, using Equation (3.6), write

$$(s-\sigma)\ \mathrm{row}_i(Q)\cdot[f_s]_{\mathrm{Kb}} = D_{i,i}\ \mathrm{row}_i(P^{-1})\cdot[f_s]_{\mathrm{Kb}}.$$

We now explain how to compute the projection $e_{\text{eigenspace }\sigma}(H)$ of an overconvergent modular form H in $M_k^{\mathrm{oc}}(\mathbb{Z}_p, N, \chi; \frac{p}{p+1})$ over the σ-eigenspace. First, we know that

$$H = \rho f_\sigma + \sum_{s\neq\sigma} F_s, \tag{3.10}$$

for some constant ρ, since we are assuming that the σ-eigenspace is one dimensional. This gives us $\pi\cdot[H] = \rho(\pi\cdot[f_\sigma])$. This is why we call π the *projector* to f_σ. Now, since

π is non trivial, it cannot be orthogonal to all modular forms, so $\pi \cdot [f_\sigma]$ cannot also be zero. We hence deduce the following formula for the projection of H over f_σ:

$$\lambda_{f_\sigma}(H) := \rho = \frac{\pi \cdot [H]_{\mathrm{Kb}}}{\pi \cdot [f_\sigma]_{\mathrm{Kb}}}. \tag{3.11}$$

More formally, the projection operator λ_{f_σ} over f_σ is the unique Hecke-equivariant linear functional that factors through the Hecke eigenspace associated to f_σ and is normalized to send f_σ to 1, as in Definition 2.7 of [Loe18] (see also Section 9.2 of [LZ16]). This gives us an associated idempotent operator $e_{f_\sigma}(\cdot) := \lambda_{f_\sigma}(\cdot)\ f_\sigma$. Since we are assuming that the σ-eigenspace is one dimensional, we have $e_{\text{eigenspace } \sigma}(H) = e_{f_\sigma}(H)$.

As explained in Remark 6, this holds under the assumption that H has growth condition $\frac{p}{p+1}$. In the case where H has growth condition $\frac{1}{p+1}$, we need to use a similar trick to that of Remark 6. We first apply the Atkin operator to H to obtain a modular form $U_p(H)$ of growth rate $\frac{p}{p+1}$. Then, we follow the algorithm described above as usual. However, we need to adjust our final output by dividing by σ to compensate for the fact that we took $U_p(H)$ instead of H.

Indeed, write H as a sum $H = \rho f_\sigma + \sum_{s \neq \sigma} F_s$, as in Equation (3.10). Then,

$$\begin{aligned} U_p(H) &= \rho\ U_p(f_\sigma) + \sum_{s \neq \sigma} U_p(F_s) \\ &= \rho \sigma f_\sigma + \sum_{s \neq \sigma} U_p(F_s). \end{aligned}$$

Since the action of U_p preserves the eigenspaces of $M_k^{\mathrm{oc}}(N)$, we get that $\pi \cdot [U_p(F_s)]_{\mathrm{Kb}} = 0$ for $s \neq \sigma$, so $\pi \cdot [U_p(H)]_{\mathrm{Kb}} = \rho\sigma\ \pi \cdot [f_\sigma]_{\mathrm{Kb}}$. Finally,

$$\lambda_{f_\sigma}(U_p(H)) = \rho\sigma = \sigma\ \lambda_{f_\sigma}(H).$$

We thus obtain

$$\lambda_{f_\sigma}(H) = \frac{\pi \cdot [U_p(H)]_{\mathrm{Kb}}}{\sigma\ \pi \cdot [f_\sigma]_{\mathrm{Kb}}}. \tag{3.12}$$

3.2.2 The case of multiplicity greater than 1

In the case where the eigenvalue σ has multiplicity r greater than one, the eigenspace associated to σ will contain eigenforms other than the one we are projecting on. The method we are presenting here will thus not work because the projector $e_{\text{eigenspace } \sigma}$ over σ-eigenspace is not equal to e_{f_σ} anymore. In this case, one needs to use the last r rows of Q and the other Hecke operators in order to find a system of equations to solve and obtain $\lambda_{f_\sigma}(H)$.

As a simple example, assume that we already have a basis for the σ-eigenspace consisting of normalized Hecke eigenforms $\{\mathfrak{f}_1, ..., \mathfrak{f}_r\}$, with $\mathfrak{f}_1 = f_\sigma$. We then express the eigenspace σ projection of H as a linear combination $\sum_j a_j \mathfrak{f}_j$. Using the last r rows $\pi_1, ..., \pi_r$ of Q, we obtain a system of equations $\pi_i \cdot [H] = \sum_j a_j\ \pi_i \cdot [\mathfrak{f}_j]$. This can easily be solved in order to find $a_1 = \lambda_{f_\sigma}(H)$. The author has not yet implemented this method.

3.2.3 Stabilizations of Hecke eigenforms

We end Section 3.2, which mainly discusses the issue of eigenspace σ projections, by explaining how one can construct a modular form f_σ of eigenvalue σ and slope $\mathrm{val}_p(\sigma)$, for certain values of σ. This method will be quite useful later on in this thesis for two reasons. Firstly, it will reappear in the definition of the p-adic Garrett-Rankin triple product L-function (in Section 4.1). Secondly, it will give us modular forms of eigenvalue σ with which we will be able to test our eigenspace σ projections algorithms.

Let $p \nmid N$. Consider an overconvergent modular form $f = \sum a_n q^n$ of weight k and level N. Assume that f is an eigenform for the operator T_p,

$$T_p f = a_p f. \tag{3.13}$$

We can then define two new eigenforms f_α and f_β as follows. Let α, β be the roots of the *Hecke polynomial*

$$x^2 - a_p x + p^{k-1}\chi(p). \tag{3.14}$$

Assume that the modular form f is *regular at* p, i.e. that α and β are different. We sometimes denote α and β by $\alpha_{f,p}$ and $\beta_{f,p}$ to emphasize the dependence on f and p. Assume as well that f is *ordinary at* p, i.e. that one of the roots of $x^2 - a_p x + p^{k-1}\chi(p)$ is a p-adic unit (i.e. has valuation zero). Say without loss of generality that $\mathrm{val}_p(\alpha_{f,p}) = 0$. Define the following two modular forms:

$$f_\alpha(q) := f(q) - \beta f(q^p);$$

$$f_\beta(q) := f(q) - \alpha f(q^p).$$

We call f_α and f_β the *p-stabilizations* of f. They both have level pN. Since we assumed that $\alpha_{f,p}$ is a unit, it is customary to call $f_\alpha(q) := f(q) - \beta f(q^p)$ the *ordinary p-stabilization* of f. Using equation (3.13) and the fact that $V = \mathrm{Frob}_p$ is a right inverse of U_p, we write

$$\begin{aligned} U_p(f_\alpha) &= U_p(f) - \beta(U_p V)f \\ &= a_p f - \chi(p)p^{k-1}Vf - \beta f \\ &= (\alpha+\beta)f - \alpha\beta V f - \beta f \\ &= \alpha f_\alpha. \end{aligned}$$

Similarly,

$$U_p(f_\beta) = \beta f_\beta.$$

Hence, f_α (respectively, f_β) is an eigenvalues of U_p with eigenvalue α (respectively, β).

In the rest of this thesis, we will be interested in taking the form $H := \mathrm{d}^{-(1+t)}(g^{[p]}) \times h$ (see Section 3.1.3) and compute its projection over the eigenspaces generated by f_α and f_β. We will do so using the methods described in this section. The following section introduces the theoretical setting that motived us to work on eigenspace σ projections of p-adic modular forms.

Chapter 4

A p-adic symbol for triples of modular forms

This chapter is the second main component of this thesis. In it we introduce a new p-adic symbol for triples of modular forms. This new theoretical object is interesting in its own right, as it is related to a known triple product p-adic L-function, in addition to enjoying nice symmetry properties. Moreover, our new p-adic symbol will give us the perfect examples to use the new algorithms described in Chapter 3.

In Section 3.1.3, we described how to compute the ordinary projection of the nearly overconvergent modular form

$$H := \mathrm{d}^{-(1+t)}(g^{[p]}) \times h,$$

where g, h are two classical modular forms of weights ℓ, m respectively. In [DR14], the authors consider the same H, but require that $t = \ell + m - 2$ to ensure that H is overconvergent. We will work in greater generality here and allow $0 \leq m \leq \min\{k, \ell\} - 2$. The p-adic modular form H will still be nearly overconvergent. The reason why we wanted to compute this H specifically is that it appears in the expression of the Garrett-Rankin triple product p-adic L-function $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m)$, in Definition 4.1.1.

4.1 The Garrett-Rankin triple product p-adic L-function

Let f, g, h be three cuspidal eigenforms of level N, respective weights k, ℓ, m and respective characters χ_f, χ_g, χ_h. Fix a prime $p \geq 5$ and assume that $p \nmid N$, that $\chi_f \chi_g \chi_h = 1$ and that the weights k, ℓ, m are balanced, i.e. the largest one is strictly smaller than the sum of the other two. Note that the assumption that $p \geq 5$ is purely for simplicity and could potentially be relaxed at the cost of some extra care. Let α_f, β_f be the roots of the *Hecke polynomial* (3.14) associated to f. Assume that the modular forms f, g and h are *ordinary* and *regular* at p, so that $\alpha_{f,p}, \alpha_{g,p}$ and $\alpha_{h,p}$ are units. Consider the the p-stabilizations f_α and f_β of f. They both have level pN, and are eigenforms for the U_p

operator with respective eigenvalues α_f and β_f. Let

$$t := \frac{\ell + m - k - 2}{2} \geq 0, \qquad c := \frac{k + \ell + m - 2}{2}.$$

We may now define the Euler factors:

$$\begin{aligned}
\mathcal{E}(f,g,h) &:= (1-\beta_f\alpha_g\alpha_h p^{-c})(1-\beta_f\alpha_g\beta_h p^{-c})(1-\beta_f\beta_g\alpha_h p^{-c})(1-\beta_f\beta_g\beta_h p^{-c});\\
\tilde{\mathcal{E}}(f,g,h) &:= (1-\alpha_f\alpha_g\alpha_h p^{-c})(1-\alpha_f\alpha_g\beta_h p^{-c})(1-\alpha_f\beta_g\alpha_h p^{-c})(1-\alpha_f\beta_g\beta_h p^{-c});\\
\mathcal{E}_0(f) &:= 1-\beta_f^2\chi_f^{-1}(p)p^{1-k}; \qquad \tilde{\mathcal{E}}_0(f) := 1-\alpha_f^2\chi_f^{-1}(p)p^{1-k};\\
\mathcal{E}_1(f) &:= 1-\beta_f^2\chi_f^{-1}(p)p^{-k}; \qquad \tilde{\mathcal{E}}_1(f) := 1-\alpha_f^2\chi_f^{-1}(p)p^{-k}.
\end{aligned} \tag{4.1}$$

Following Section 2.6 of [DR14], let $\Gamma := 1 + pN\mathbb{Z}_p$, and let $\Lambda := \mathcal{O}[[\Gamma]]$ be the completed group ring of Γ. Let also $\Lambda' := \mathrm{Frac}(\Lambda)$. Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be Hida families, with coefficients in finite flat extensions $\Lambda_f, \Lambda_g, \Lambda_h$ of Λ, interpolating f, g and h at the weights k, ℓ and m. The existence of such families is guaranteed by Hida's construction in [Hid86]. Let $\mathbf{f}^* := \mathbf{f} \otimes \chi_f^{-1}$, and note that for classical points x (in $\mathbb{Z}$) we have $(f^*)_x = (f_x)^*$. We write $\kappa(x)$ for the weight of f_x.

Given an ordinary eigenform F and an ordinary overconvergent modular form G (for example $e_{\mathrm{ord}}(\mathrm{d}^{-1-t}(g_y{}^{[p]}) \times h_z)$ from (4.2)), we introduce the operator $c(F, G)$. It denotes the coefficient of F appearing in the expression of G as a linear combination of ordinary (normalized) eigenforms (see [Hid93], p. 222).

Remark 10. In order to define the pairing $c(F, G)$, we are assuming here that the action of the Hecke algebra on the ordinary subspace in weight k is semi-simple (see assumption $(S3)$ on p. 222 in [Hid93]). This is the case for N square-free, since $k \geq 2$, as is described in [Lau14].

Definition 4.1.1 (Lemma 2.19, [DR14])**.** The Garrett-Rankin triple product p-adic L-function attached to the triple $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of Λ-adic modular forms is the unique $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ in $\Lambda'_f \otimes_\Lambda (\Lambda_g \otimes \Lambda_h \otimes \Lambda)$ such that at classical balanced points (x, y, z) we have

$$\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z) := c\left(f_x^{*(p)}, e_{\mathrm{ord}}(\mathrm{d}^{-1-t}(g_y{}^{[p]}) \times h_z)\right), \tag{4.2}$$

where $t := \frac{\kappa(y)+\kappa(z)-\kappa(x)-2}{2}$, $f_x^* := f_x \otimes \chi_f^{-1}$ is the dual of f_x and $f_x^{*(p)}$ is the ordinary p-stabilization of f_x^*. We write

$$\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h}) := c\left(\mathbf{f}^*, e_{\mathrm{ord}}(\mathrm{d}^\bullet \mathbf{g}^{[p]} \times \mathbf{h})\right)$$

for notational brevity.

Remark 11. The Garrett-Rankin triple product p-adic L-function inherits its name from the classical Garrett-Rankin triple product L-function (cf. [Gar87, PSR87]), as Darmon and Rotger have shown that the former interpolates certain values related the latter (see Remark 4.8 in [DR14]).

Note that in [DR14], the authors use the notation $\mathscr{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$. And when one wishes to project on the g component instead of the f component, they introduced the notation $\mathscr{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathscr{L}_p^g(\mathbf{g}, \mathbf{h}, \mathbf{f})$. We use here the more compact notation $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$. And for the projection on the g component, we simply write $\mathscr{L}_p(\mathbf{g}, \mathbf{h}, \mathbf{f})$. Thus we always project on the first component appearing amongst the three used ones.

Moreover, we use $\mathbf{f}^*$ instead of $\mathbf{f}$ in Definition 4.1.1. This is because $e_{\mathrm{ord}}(\mathrm{d}^\bullet \mathbf{g}^{[p]} \times \mathbf{h})$ has character $\chi_g\chi_h = \chi_f^{-1}$, hence the need to project over $\mathbf{f}^* := \mathbf{f} \otimes \chi_f^{-1}$ and not $\mathbf{f}$.

Given a cuspidal newform ϕ, let λ_ϕ be the projection operator over ϕ; it is the unique Hecke-equivariant linear functional that factors through the Hecke eigenspace associated to ϕ and is normalized to send ϕ to 1, as in Definition 2.7 of [Loe18]. This allows to express the Garrett-Rankin triple product p-adic L-function $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at classical balanced points (x, y, z) as

$$\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z) := \lambda_{f^*_{x,\alpha}}(d^{-1-t} g_y^{[p]} \times h_z), \tag{4.3}$$

where $f^*_{x,\alpha} := (f^*_x)_\alpha$ is the ordinary p-stabilization of the dual of f_x.

In order to experimentally compute the values of $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z)$, Equation (4.2) reveals that the main ingredient is the computation of ordinary projections of p-adic modular forms. In [Lau14], parts of which have been summarized here in Section 3.1.2, the author explains how to calculate the ordinary projections of overconvergent modular forms, and is thus able to compute special values of the Garrett-Rankin triple product p-adic L-function, for balanced weights (k, ℓ, m) satisfying $k = 2 + m - \ell$. Indeed, this condition guarantees that $\mathrm{d}^{-1-t}({g_\ell}^{[p]}) \times h_m$ will be overconvergent, thus the code and the theory in [Lau14] are enough.

In general, however, when the weights (k, ℓ, m) are only balanced, $\mathrm{d}^{-1-t}({g_y}^{[p]}) \times h_z$ is only *nearly overconvergent*. We therefore need to use the generalizations we introduced in Section 3.1.3 in order to compute ordinary projections of nearly overconvergent modular forms, thus being able to compute the Garrett-Rankin triple product p-adic L-function for any balanced classical weights.

In Section 3 of [DR14], the authors construct a generalized Gross-Kudla-Schoen diagonal cycle $\Delta := \Delta_{k,\ell,m}$ for a triple of balanced classical weights (k, ℓ, m). More precisely, this cycle is an element of the Chow group $\mathrm{CH}^{r+2}(W)_0$ where $W := \mathcal{E}^{k-2} \times \mathcal{E}^{\ell-2} \times \mathcal{E}^{m-2}$ and $r := (k + \ell + m)/2 - 3$ (cf. Chapter 1 in [Ful13] for more on Chow groups). One can check from Definition 3.3 of [DR14] that $\Delta_{k,\ell,m}$ indeed has codimension $r + 2$. Let

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(W)_0 \longrightarrow \mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W)^\vee. \tag{4.4}$$

be the p-adic Abel-Jacobi map (cf. Section (1.2) of [Nek00] or [Bes00]). Darmon and Rotger then show, in Theorem 3.14 of [DR14], that

$$\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h) = (-1)^{t+1} t! \frac{\mathcal{E}_1(f)}{\mathcal{E}(f, g, h)} \langle \eta_f^{\text{u-r}}, d^{-1-t} g^{[p]} \times h \rangle, \tag{4.5}$$

where $t := \frac{\ell+m-k-2}{2}$. In Equation (4.5), $\omega_g \in H^{\ell-1}_{\mathrm{dR}}(\mathcal{E}^{\ell-2}/\mathbb{C}_p)_g$ and $\omega_h \in H^{m-1}_{\mathrm{dR}}(\mathcal{E}^{m-2}/\mathbb{C}_p)_h$ denote the differentials associated to the forms g and h respectively, that we introduced at the start of Section 2.3. Furthermore, $\eta_f^{\text{u-r}}$ will denote the element lying in the unit root space of $H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{k-2}/\mathbb{C}_p)_{f^*}$ such that $\langle \omega_f, \eta_f^{\text{u-r}} \rangle = 1$ (see the discussion in Section 2.3), where $\omega_f \in H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{k-2}/\mathbb{C}_p)_{f^*}$ is the differential associated to f^*. Note that although our notation for ω_g and ω_h is the same as the one introduced at the start of Section 2.3, our notation for ω_f and $\eta_f^{\text{u-r}}$ is not. Indeed, the roles of f and f^* have been switched in ω_f and $\eta_f^{\text{u-r}}$; but g and h are still the same and have not been replaced by their duals in ω_g and ω_h. This choice is necessary, as explained in the second part of Remark 11, and is consistent with the notation used in [DR14]. Finally, given the cohomology classes $\eta_f^{\text{u-r}} \in H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{k-2})$, $\omega_g \in H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{\ell-2})$ and $\omega_h \in H^{k-1}_{\mathrm{dR}}(\mathcal{E}^{m-2})$, we can view the product $\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h$ in Equation (4.5) as an element of $\mathrm{H}^{2r+3}_{\mathrm{dR}}(W)$ thanks to the Künneth decomposition (cf. [Kün23, Kün24]).

We now, as in Theorem 5.1 of [DR14], provide an alternative way to express the Garrett-Rankin triple product p-adic L-function by relating it to the generalized Gross-Kudla-Schoen diagonal cycle as follows.

Proposition 4.1.2. *We have*

$$\mathrm{AJ}_p(\Delta)(\eta_f^{u\text{-}r} \otimes \omega_g \otimes \omega_h) = (-1)^t t! \frac{\mathcal{E}_0(f)\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \lambda_{f_\alpha^*}(d^{-1-t} g^{[p]} \times h).$$

Proof. By Theorem 3.14 in [DR14], we have

$$\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h) = \left\langle \eta_f^{\text{u-r}}, -\frac{(-1)^t t! \mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} e_{f^*,\text{ord}}(d^{-1-t} g^{[p]} \times h) \right\rangle.$$

Note that we write $\langle \eta_f^{\text{u-r}}, \phi \rangle$ here to mean $\langle \eta_f^{\text{u-r}}, \omega_\phi \rangle$ by abuse of notation. We observe that the f^*-isotypic component of $e_{\text{ord}}(d^{-1-t} g^{[p]} \times h)$ is $\lambda_{f_\alpha^*}(d^{-1-t} g^{[p]} \times h) f_\alpha^*$, because we can express $e_{\text{ord}}(d^{-1} g^{[p]} \times h)$ as

$$\lambda_{f_\alpha^*}(d^{-1-t} g^{[p]} \times h) f_\alpha^* + (\text{terms attached to other ordinary forms}).$$

Therefore,

$$\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h) = (-1)^{t+1} t! \frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \lambda_{f_\alpha^*}(d^{-1-t} g^{[p]} \times h) \left\langle \eta_f^{\text{u-r}}, f_\alpha^* \right\rangle.$$

Next, $f_\alpha^* = \mathcal{E}_0(f) e_{\text{ord}}(f^*)$ by applying the proof of Lemma 4.2.1 to f^* instead of f, thus by Proposition 2.11 in [DR14],

$$\left\langle \eta_f^{\text{u-r}}, f_\alpha^* \right\rangle = \mathcal{E}_0(f) \left\langle \eta_f^{\text{u-r}}, e_{\text{ord}}(f^*) \right\rangle = \mathcal{E}_0(f) \left\langle \eta_f^{\text{u-r}}, f^* \right\rangle = -\mathcal{E}_0(f),$$

as $\left\langle \eta_f^{\text{u-r}}, f^* \right\rangle = -\left\langle f^*, \eta_f^{\text{u-r}} \right\rangle = -1$ by definition of $\eta_f^{\text{u-r}}$. We finally obtain

$$\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h) = (-1)^t t! \frac{\mathcal{E}_0(f)\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \lambda_{f_\alpha^*}(d^{-1-t} g^{[p]} \times h).$$

□

Equation (4.3) together with Proposition 4.1.2 give the following alternative expression for the Garrett-Rankin triple product p-adic L-function.

Corollary 4.1.3. *The Garrett-Rankin triple product p-adic L-function can be written, at classical balanced points (k, ℓ, m), as*

$$\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m) = \frac{(-1)^t}{t!} \frac{\mathcal{E}(f,g,h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \mathrm{AJ}_p(\Delta)(\eta_f^{u\text{-}r} \otimes \omega_g \otimes \omega_h), \tag{4.6}$$

Equation (4.2), or equivalently Equation (4.3), provides us with an elegant and compact way to express the Garrett-Rankin p-adic L-function at classical balanced points. Equation (4.6) on the other hand connects it to the Abel Jacobi map and provides us with the right insight in order to define a new, more natural, version of the Garrett-Rankin triple product p-adic L-function, which we expect to have nice symmetry properties.

4.2 A new p-adic triple symbol $(f, g, h)_p$

We continue working in the same setup as the previous section. The differentials $\omega_g \in H^{\ell-1}_{\mathrm{dR}}(\mathcal{E}^{\ell-2}/\mathbb{C}_p)_g$ and $\omega_h \in H^{m-1}_{\mathrm{dR}}(\mathcal{E}^{m-2}/\mathbb{C}_p)_h$ are the basis elements that we introduced at the start of Section 2.3. Similarly to Section 4.1, ω_f will denote the differential associated to f^*, and not f. This choice is necessary, as explained in the second part of Remark 11.

Our goal is to define a new quantity involving $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ instead of $\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h)$, and we believe that this alternative should have nice symmetry properties. We investigate such properties further in the remaining sections.

When computing $\mathrm{AJ}_p(\Delta)(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h)$, we only need to consider ordinary projections, i.e. projections on the slope 0 subspace, as in [DR14] and [Lau14]. In the case of $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ however, as we will see in Lemma 4.2.2, we need to compute projections on the slope $k-1$ subspace as well. More specifically, we will need to compute two eigenspace projections for certain prescribed eigenvalues, as is explained below. This can efficiently be done using the algorithm described in Section 3.2.

Before defining a new p-adic triple symbol, we first provide a way to express $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ in terms of projections onto isotypic spaces, similarly to Proposition 4.1.2. Let

$$\ell_{fgh,\alpha} := \lambda_{f^*_\alpha}\left(\mathrm{d}^{-1-t}(g^{[p]}) \times h\right); \qquad \ell_{fgh,\beta} := \lambda_{f^*_\beta}\left(\pi_{\mathrm{oc}}\left(\mathrm{d}^{-1-t}(g^{[p]}) \times h\right)\right). \tag{4.7}$$

Note that including π_{oc} before $\lambda_{f^*_\alpha}$ in (4.7) would be redundant, by Theorem 2.2.3. We have the following first attempt at making the quantity $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ a little more tractable.

Lemma 4.2.1. *Let f be a classical eigenform of weight k that is ordinary at p with $\mathrm{val}_p(\alpha_{f,p}) = 0$. Then, we have $e_{ord}(f) = \frac{1}{\mathcal{E}_0(f)} f_\alpha$ and $e_{slope\ k-1}(f) = \frac{1}{\tilde{\mathcal{E}}_0(f)} f_\beta$.*

Proof. We have by definition $f_\alpha(q) := f(q) - \beta f(q^p)$ and $f_\beta(q) := f(q) - \alpha f(q^p)$. So, $\alpha f_\alpha - \alpha f = \beta f_\beta - \beta f$. Hence, $\alpha f_\alpha - \beta f_\beta = (\alpha - \beta) f$. Thus, using the notation from (4.1), we have

$$e_{\mathrm{ord}}(f) = \frac{\alpha f_\alpha}{\alpha - \beta} = \frac{1}{\mathcal{E}_0(f)} f_\alpha, \qquad e_{\text{slope } k-1}(f) = \frac{\beta f_\beta}{\beta - \alpha} = \frac{1}{\tilde{\mathcal{E}}_0(f)} f_\beta.$$

□

Lemma 4.2.1 shows that in order for us to compute slope 0 and $k-1$ projections of f, we only need to compute eigenspace α and β projections of f, respectively. This explains why we only turned our attention to eigenspace projections in Section 3.2.

Lemma 4.2.2. *Let $t := \frac{\ell+m-k-2}{2}$. We may rewrite $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ as*

$$(-1)^t t! \left(\frac{\mathcal{E}_0(f)\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \ell_{fgh,\alpha} \langle \omega_f, e_{ord}(f^*) \rangle + \frac{\tilde{\mathcal{E}}_0(f)\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \ell_{fgh,\beta} \langle \omega_f, e_{slope\ k-1}(f^*) \rangle \right).$$

Proof. Note that f^* is orthogonal to the kernel of e_{f^*}, so $\langle f^*, \phi \rangle = \langle f^*, e_{f^*}(\phi) \rangle$ only depends on the projection $e_{f^*}(\phi)$ of ϕ, for any modular form ϕ. Adapting this to our notation, we obtain $\langle \omega_f, \phi \rangle = \langle \omega_f, e_{f^*}(\phi) \rangle$, as ω_f, here, is the differential attached to

f^*. Furthermore, $e_{f^*}(\phi)$ only depends on the overconvergent projection of ϕ. Indeed, $\phi - \pi_{\mathrm{oc}}(\phi)$ is purely nearly overconvergent (i.e. it has no overconvergent part) and will not lie in the f^*-isotypic space, as f^* is overconvergent. Lemma 4.2.1 tells us that f has only two slope components: an ordinary one and one of slope $k-1$. Namely, $f = \frac{1}{\mathcal{E}_0(f)} f_\alpha + \frac{1}{\tilde{\mathcal{E}}_0(f)} f_\beta$, and thus to project over the f^*-isotypic space, one needs to project over the components f_α^* and f_β^*. Adapting the proof of Proposition 4.1.2 for the case of $\mathrm{AJ}(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$, and using the notation $\xi(\omega_g, \omega_h)$ from [DR14] (see Equation (72) on p. 30), we write

$$\begin{aligned}
\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h) &= \langle \omega_f, \xi(\omega_g, \omega_h) \rangle \\
&= \langle \omega_f, e_{f^*,\mathrm{ord}}(\xi(\omega_g, \omega_h)) + e_{f^*,\mathrm{slope}\ k-1}(\xi(\omega_g, \omega_h)) \rangle \\
&= \left\langle \omega_f, -\frac{(-1)^t t! \mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} e_{f^*,\mathrm{ord}}(d^{-1-t} g^{[p]} \times h) \right\rangle \\
&\quad + \left\langle \omega_f, -\frac{(-1)^t t! \tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} e_{f^*,\mathrm{slope}\ k-1}\left(\pi_{\mathrm{oc}}(d^{-1-t} g^{[p]} \times h)\right) \right\rangle \\
&\overset{(\star)}{=} -(-1)^t t! \frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \lambda_{f_\alpha^*}(d^{-1-t} g^{[p]} \times h) \langle \omega_f, f_\alpha^* \rangle \\
&\quad - (-1)^t t! \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \lambda_{f_\beta^*}\left(\pi_{\mathrm{oc}}(d^{-1-t} g^{[p]} \times h)\right) \langle \omega_f, f_\beta^* \rangle \qquad (4.8) \\
&= -(-1)^t t! \frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \lambda_{f_\alpha^*}(d^{-1-t} g^{[p]} \times h) \mathcal{E}_0(f) \langle \omega_f, e_{\mathrm{ord}}(f^*) \rangle \\
&\quad - (-1)^t t! \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \lambda_{f_\beta^*}\left(\pi_{\mathrm{oc}}(d^{-1-t} g^{[p]} \times h)\right) \tilde{\mathcal{E}}_0(f) \langle \omega_f, e_{\mathrm{slope}\ k-1}(f^*) \rangle \\
&= -(-1)^t t! \frac{\mathcal{E}_0(f)\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \ell_{fgh,\alpha} \langle \omega_f, e_{\mathrm{ord}}(f^*) \rangle \\
&\quad - (-1)^t t! \frac{\tilde{\mathcal{E}}_0(f)\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \ell_{fgh,\beta} \langle \omega_f, e_{\mathrm{slope}\ k-1}(f^*) \rangle .
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h) = (-1)^{t+1} t! \Bigg(&\frac{\mathcal{E}_0(f)\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \ell_{fgh,\alpha} \langle \omega_f, e_{\mathrm{ord}}(f^*) \rangle \\
&+ \frac{\tilde{\mathcal{E}}_0(f)\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \ell_{fgh,\beta} \langle \omega_f, e_{\mathrm{slope}\ k-1}(f^*) \rangle \Bigg),
\end{aligned}$$

as required. □

At this point, we remark that the quantity obtained in Lemma 4.2.2 doesn't seem to be easily computable in any obvious way; in particular, we aren't sure how to directly compute $\langle \omega_f, e_{\mathrm{ord}}(f) \rangle$ and $\langle \omega_f, e_{\mathrm{slope}\ k-1}(f) \rangle$. To get around this issue, we go back to step $(\star)$ in Equation (4.8), From there, we go in a different direction, to find an alternative expressions for $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$.

Theorem 4.2.3. *Let* $t := \frac{\ell+m-k-2}{2}$. *We have*

$$\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h) = (-1)^t t! \frac{\langle \omega_f, \phi(\omega_f) \rangle}{p^{k-1}} \left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \ \beta_{f^*} \ \ell_{fgh,\alpha} + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \ \alpha_{f^*} \ \ell_{fgh,\beta} \right).$$

Proof. First, note that the action of Frobenius on ω_f is given by $\phi(\omega_f) = \omega_{Vf^*}$. Indeed, when f has weight 2, this can bee seen by directly computing the action of Frobenius on q-expansions and using Equation (2.15). For higher weights, this is explained in the proof of Lemma 2.10 in [DR14]. Now, as $\langle \omega_f, f^* \rangle = \langle \omega_f, \omega_f \rangle = 0$, we can write

$$\begin{aligned}
\langle \omega_f, f^*_\alpha \rangle &= \langle \omega_f, f^* - \beta_{f^*} V f^* \rangle \\
&= \langle \omega_f, f^* \rangle - \beta_{f^*} \langle \omega_f, V f^* \rangle \\
&= -\beta_{f^*} \langle \omega_f, \omega_{Vf^*} \rangle \\
&= -\frac{\beta_{f^*}}{p^{k-1}} \langle \omega_f, \phi(\omega_f) \rangle .
\end{aligned}$$

Similarly, $\langle \omega_f, f^*_\beta \rangle = -\frac{\alpha_{f^*}}{p^{k-1}} \langle \omega_f, \phi(\omega_f) \rangle$. Substituting this into step $(\star)$ of Equation (4.8), we obtain

$$\begin{aligned}
\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h) &= -(-1)^t t! \frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \ell_{fgh,\alpha} \langle \omega_f, f^*_\alpha \rangle \\
&\quad - (-1)^t t! \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \ell_{fgh,\beta} \langle \omega_f, f^*_\beta \rangle \\
&= -(-1)^t t! \frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \frac{-\beta_{f^*}}{p^{k-1}} \ell_{fgh,\alpha} \langle \omega_f, \phi(\omega_f) \rangle \\
&\quad - (-1)^t t! \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \frac{-\alpha_{f^*}}{p^{k-1}} \ell_{fgh,\beta} \langle \omega_f, \phi(\omega_f) \rangle ,
\end{aligned}$$

which we rewrite as

$$\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h) = (-1)^t t! \frac{\langle \omega_f, \phi(\omega_f) \rangle}{p^{k-1}} \left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f,g,h)} \ \beta_{f^*} \ \ell_{fgh,\alpha} + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f,g,h)} \ \alpha_{f^*} \ \ell_{fgh,\beta} \right).$$

□

We can compute all the terms appearing in the last line of (4.5). Indeed, the calculation of $\mathcal{E}_1(f), \mathcal{E}(f,g,h), \tilde{\mathcal{E}}_1(f), \tilde{\mathcal{E}}(f,g,h)$ is a straightforward application of a formula. The constants α_f, β_f are directly obtained by factoring the quadratic Hecke polynomial (3.14). We can compute $\ell_{fgh,\alpha}, \ell_{fgh,\beta}$ by using the method described in Section 3.2. Finally, we can compute the period $\langle \omega_f, \phi(\omega_f) \rangle$ by using Kedlaya's algorithm (cf. [Ked01]), in the case where f has weight 2. In the other cases, we find work arounds.

We are now ready to write down our p-adic symbol for triples of modular forms. Initially, one would think to mimic the definition of $\mathscr{L}_p(f,g,h)$ but replace $\eta_f^{\text{u-r}}$ by ω_f, thus obtaining the following contender

$$\frac{(-1)^t}{t!} \frac{\mathcal{E}(f,g,h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} \mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h), \tag{4.9}$$

for the definition of $(f,g,h)_p$. However, experimental evidence, which we will present in Chapter 5, shows that $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ seems to already be symmetric (when cyclically permuting f,g,h). Therefore, adding the factor $\frac{\mathcal{E}(f,g,h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)}$ to $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ would ruin the symmetry. Note that $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ already has correction factors in it, as we can see in Theorem 4.2.3.

Definition 4.2.4. Let f, g and h be three cuspidal modular forms of level N and weight k, ℓ and m (respectively) which are ordinary at p. We define the p-adic triple symbol $(f, g, h)_p$ by

$$(f, g, h)_p := (-1)^t t! \frac{\langle \omega_f, \phi(\omega_f) \rangle}{p^{k-1}} \left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f, g, h)} \beta_{f^*} \ell_{fgh,\alpha} + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f, g, h)} \alpha_{f^*} \ell_{fgh,\beta} \right). \quad (4.10)$$

In Definition 4.2.4, we do not actually need g and h to be cuspidal nor ordinary at p, as Equation (4.10) is still defined when only f is. However, as we are interested in permuting the order of f, g and h, we often require them all to be cuspidal and ordinary at p. Thanks to Theorem 4.2.3, we may reformulate $(f, g, h)_p$ as follows.

Corollary 4.2.5. *We can express the p-adic triple symbol $(f, g, h)_p$ as*

$$(f, g, h)_p := \mathrm{AJ}_p(\Delta_{k,\ell,m})(\omega_f \otimes \omega_g \otimes \omega_h). \quad (4.11)$$

The right hand side of Equation (4.11) appears to be symmetric in the variables f, g, h, and thus suggests that $(f, g, h)_p$ is symmetric.

4.3 Symmetry properties of $(f, g, h)_p$

There are two types of symmetry that can be considered for $(f, g, h)_p$. The first, and the easiest to prove, is the one between the second and third variables of $(f, *, *)_p$ when we fix f. The second kind is the one between all three variable of $(f, g, h)_p$ (or equivalently the one between the first and second variables). That symmetry is more interesting, involves deeper mathematics.

4.3.1 Partial symmetry for $(f, *, *)_p$

We will explain how we were lead to proving such a symmetry result by describing the path the author took to finally obtain the desired proof, starting with the experimental computations. We will therefore emphasize the close connection between computational number theory and the theory – especially since we have gone through the trouble of developing robust algorithms in Chapter 3 for computing projections of p-adic modulars forms, therefore allowing us to compute $(f, g, h)_p$.

4.3.1.1 Computational evidence

Our experiments have shown that in some cases $(f, g, h)_p = (f, h, g)_p$; while in some other cases $(f, g, h)_p = -(f, h, g)_p$. We will present here some of the experimental results we obtained, and summarize the whole of the data in a table further below. In what follows, we will only present values of $(f, g, h)_p$ but not explain how the computations are performed, as this would makes this subsection quite longer, and distract us from the task at hand (understanding the behaviour of $(f, *, *)_p$). We will present multiple detailed examples of the calculation of $(f, g, h)_p$ in Chapter 5.

When comparing $(f,g,h)_p$ and $(f,h,g)_p$, we do not have to actually compute all the terms appearing the expansions. Indeed, looking at Equation (4.10), we see that $\Omega_f := \langle \omega_f, \phi(\omega_f)\rangle$ appears in both $(f,g,h)_p$ and $(f,h,g)_p$. So we can only consider $(f,g,h)_p/\Omega_f$ and $(f,h,g)_p/\Omega_f$, when studying the symmetry of $(f,*,*)_p$. This has the advantage of avoiding the Poincaré pairing $\langle \omega_f, \phi(\omega_f)\rangle$, which is quite mysterious and not computable for weights greater than 2.

Weight $(4,6,6)$**:** Let $f_1, f_2, \in S_6^{\text{new}}(10,\mathbb{Q})$ and $f \in S_4^{\text{new}}(10,\mathbb{Q})$ be the modular forms with MAGMA labels "G0N10k6A","G0N10k6B" and "G0N10k4A", respectively. Then,

$$(f_1, f, f_2)_{11}/\Omega_{f_1} = -11574471190059869210906134237854826 \cdot 11 \bmod 11^{34}$$
$$(f_1, f_2, f)_{11}/\Omega_{f_1} = 4740719931348921631946347271949109 \cdot 11 \bmod 11^{34}.$$

And we can check that $(f_1, f, f_2)_{11}/\Omega_{f_1} = (f_1, f_2, f)_{11}/\Omega_{f_1} \bmod 11^{32}$, making $(f_1,*,*)_{11}$ symmetric.

Weight $(4,6,8)$**:** Let $f \in S_4^{\text{new}}(15,\mathbb{Q})$, $\varphi_1, \varphi_2 \in S_6^{\text{new}}(15,\mathbb{Q})$ and $\psi_1, \psi_2 \in S_8^{\text{new}}(15,\mathbb{Q})$ be the modular forms corresponding to the MAGMA labels "G0N15k4B","G0N15k6A", "G0N10k6B","G0N15k8A" and "G0N15k8B", respectively. Then,

$$(f, \varphi_1, \psi_1)_7/\Omega_f = 663668081530597986871543122130840112 \cdot 7^3 \bmod 7^{45}$$
$$(f, \psi_1, \varphi_1)_7/\Omega_f = -663668081530597986871543122130840112 \cdot 7^3 \bmod 7^{45}$$
$$(f, \varphi_2, \psi_2)_7/\Omega_f = 766431270275884387955436074320859400 \cdot 7^3 \bmod 7^{45}$$
$$(f, \psi_2, \varphi_2)_7/\Omega_f = -766431270275884387955436074320859400 \cdot 7^3 \bmod 7^{45}.$$

And we can clearly see that $(f,*,*)_7$ is anti-symmetric.

Weight $(4,8,8)$**:** Let $f_1, f_2, f_3 \in S_8^{\text{new}}(26,\mathbb{Q})$ be the modular forms with MAGMA labels "G0N26k8A", "G0N26k8B" and "G0N26k8C", respectively. Let also $f, g \in S_4^{\text{new}}(26,\mathbb{Q})$ be the modular forms with MAGMA labels "G0N26k4A" and "G0N26k4B", respectively. Then,

$$(f, f_1, f_3)_{11}/\Omega_f = 954710532341707484954873388934977 \cdot 11^6 \bmod 11^{37}$$
$$(f, f_3, f_1)_{11}/\Omega_f = 954710532341707484954873388934977 \cdot 11^6 \bmod 11^{37}$$
$$(g, f_1, f_3)_{11}/\Omega_g = 557633061026264441330447207573199 \cdot 11^5 \bmod 11^{36}$$
$$(g, f_3, f_1)_{11}/\Omega_g = 557633061026264441330447207573199 \cdot 11^5 \bmod 11^{36}.$$

Weight $(8,8,8)$**:** Again, let $f_1, f_2, f_3 \in S_8^{\text{new}}(26,\mathbb{Q})$ be the modular forms with MAGMA labels "G0N26k8A", "G0N26k8B" and "G0N26k8C", respectively. Then,

$$(f_1, f_2, f_3)_{11}/\Omega_{f_1} = -4165658041219711421061920862226 \cdot 11^3 \bmod 11^{32}$$
$$(f_1, f_3, f_2)_{11}/\Omega_{f_1} = -4165658041219711421061920862226 \cdot 11^3 \bmod 11^{32}$$
$$(f_1, f_2, f_3)_{17}/\Omega_{f_1} = -105741928641296179447241796669245184 \cdot 11^2 \bmod 11^{31}$$
$$(f_1, f_3, f_2)_{17}/\Omega_{f_1} = -105741928641296179447241796669245184 \cdot 11^2 \bmod 11^{31}.$$

Let $g_1, g_2 \in S_8^{\text{new}}(14, \mathbb{Q})$ be the modular forms with MAGMA labels "G0N14k8A" and "G0N14k8B", respectively. Then,

$$\begin{aligned}
(g_1, g_1, g_2)_{11}/\Omega_{g_1} &= 7348831052499805625709299939 60 \cdot 11^3 \bmod 11^{32} \\
(g_1, g_2, g_1)_{11}/\Omega_{g_1} &= 734883105249980562570929993960 \cdot 11^3 \bmod 11^{32} \\
(g_1, g_1, g_2)_{13}/\Omega_{g_1} &= -838107312043403407904301847253 78 \cdot 13^4 \bmod 13^{33} \\
(g_1, g_2, g_1)_{13}/\Omega_{g_1} &= -83810731204340340790430184725378 \cdot 13^4 \bmod 13^{33}.
\end{aligned}$$

The above examples seems to suggest that the symmetry behaviour depends on the weights of f, g, h. If we denote these weights by k, ℓ and m respectively, we might hazard the guess that there is some quantity $q = q(k, \ell, m) \in \mathbb{N}$ such that

$$(f, g, h)_p = (-1)^q (f, h, g)_p.$$

Upon further reflection, we notice that, in [DR14], it is explained that the quantity $\ell_{fgh,\alpha}$ (thus $\mathscr{L}_p(f, g, h)$) is anti-symmetric when the weights are $(2, 2, 2)$. We thus are lead to believe, since $(f, g, h)_p$ is a linear combination of $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$, that both $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$ satisfy symmetry properties, not just $(f, g, h)_p$. Going back to all the above examples, and computing $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$, confirms this. Below are some examples with $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$.

Weight $(4, 6, 6)$**:** Let $f_1, f_2, \in S_6^{\text{new}}(10, \mathbb{Q})$ and $f \in S_4^{\text{new}}(10, \mathbb{Q})$ be the modular forms with MAGMA labels "G0N10k6A","G0N10k6B" and "G0N10k4A", respectively. Then,

$$\begin{aligned}
\ell_{f_1 f f_2,\alpha} &= -2638696522034988452731824194680391 4432474 \bmod 11^{40} \\
\ell_{f_1 f_2 f,\alpha} &= -26386965220349884527318241946803914432474 \bmod 11^{40} \\
\ell_{f_1 f f_2,\beta} &= -7557412593541517046459365237431139 2122346 \bmod 11^{40} \\
\ell_{f_1 f_2 f,\beta} &= -15338649341563693663055559656025430884525 \bmod 11^{40}.
\end{aligned}$$

We can clearly see that $\ell_{f_1 f f_2,\alpha} = \ell_{f_1 f_2 f,\alpha} \bmod 11^{40}$ and we can easily check that $\ell_{f_1 f f_2,\beta} = \ell_{f_1 f_2 f,\beta} \bmod 11^{31}$.

Weight $(4, 6, 8)$**:** Let $f \in S_4^{\text{new}}(15, \mathbb{Q})$, $\varphi_2 \in S_6^{\text{new}}(15, \mathbb{Q})$ and $\psi_2 \in S_8^{\text{new}}(15, \mathbb{Q})$ be the modular forms with MAGMA labels "G0N15k4B","G0N15k6B" and "G0N15k8B" respectively. Then,

$$\begin{aligned}
\ell_{f\varphi_2\psi_2,\alpha} &= -80333250194943863978507768003650204197 1026 \cdot 7^{-1} \bmod 11^{49} \\
\ell_{f\psi_2\varphi_2,\alpha} &= -48128538565585575065797970608467606463 7009 \cdot 7^{-1} \bmod 11^{49} \\
\ell_{f\varphi_2\psi_2,\beta} &= -21386881935776920049309890176514450345 0454 \cdot 7^{-1} \bmod 11^{49} \\
\ell_{f\psi_2\varphi_2,\beta} &= -65614535752235463021925822523396027134 2002 \cdot 7^{-1} \bmod 11^{49}.
\end{aligned}$$

We can check that $\ell_{f\varphi_2\psi_2,\alpha} = -\ell_{f\psi_2\varphi_2,\alpha} \bmod 11^{48}$ and that $\ell_{f\varphi_2\psi_2,\beta} = -\ell_{f\psi_2\varphi_2,\beta} \bmod 11^{42}$.

Weight $(8, 8, 8)$**:** Fix $p := 13$ and let $g_1, g_2 \in S_8^{\text{new}}(14, \mathbb{Q})$ be the modular forms with MAGMA labels "G0N14k8A" and "G0N14k8B", respectively. Then,

$$\ell_{g_1 g_1 g_2,\alpha} = -79415554464330644132416479163260607504365254 \bmod 13^{40}$$

$$
\begin{aligned}
\ell_{g_1g_2g_1,\alpha} &= -79415554464330644132416479163260607504365254 \bmod 13^{40}\\
\ell_{g_1g_1g_2,\beta} &= -18760397957165789653866436844435740458836497 \bmod 13^{40}\\
\ell_{g_1g_2g_1,\beta} &= -80710539635275621185173618178739136146929888 \bmod 13^{40}.
\end{aligned}
$$

We can clearly see that $\ell_{g_1g_1g_2,\alpha} = \ell_{g_1g_2g_1,\alpha} \bmod 13^{40}$ and we can easily check that $\ell_{g_1g_1g_2,\beta} = \ell_{g_1g_2g_1,\beta} \bmod 13^{30}$.

We thus suspect the existence of some quantity $q = q(k,\ell,m) \in \mathbb{N}$ such that

$$
\ell_{fgh,\alpha} = (-1)^q \ell_{fhg,\alpha}, \qquad \ell_{fgh,\beta} = (-1)^q \ell_{fhg,\beta}.
$$

A first natural guess would be to try $q := \frac{k+\ell+m}{2}$. This seems to work with all the above examples. However, we note that we only have considered examples with even weights so far. Indeed, when considering odd weights, we notice that $q := \frac{k+\ell+m}{2}$ fails, as below.

Weight $(3,3,4)$**:** Let χ be the Legendre symbol $\left(\frac{\cdot}{7}\right)$. Let $f \in S_3^{\text{new}}(\Gamma_1(7),\mathbb{Q},\chi)$ and $g \in S_4^{\text{new}}(\Gamma_0(7),\mathbb{Q})$ be the modular forms with MAGMA labels "G1N7k3A" and "G0N7k4A" respectively. Let $p := 5$. Then,

$$
\begin{aligned}
\ell_{ffg,\alpha} &= 15605684005197 \cdot 5^{-1} \bmod 5^{19}\\
\ell_{fgf,\alpha} &= 27049775802072 \cdot 5^{-1} \bmod 5^{19}\\
\ell_{ffg,\beta} &= -22541288651053 \cdot 5^{-1} \bmod 5^{19}\\
\ell_{fgf,\beta} &= -14911894119803 \cdot 5^{-1} \bmod 5^{19}.
\end{aligned}
$$

And we can check that $\ell_{ffg,\alpha} = \ell_{fgf,\alpha} \bmod 5^{17}$ and that $\ell_{ffg,\beta} = \ell_{fgf,\beta} \bmod 5^{17}$. Since these values are non-zero, we see that $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$ are symmetric (and not anti-symmetric) despite $q(3,3,4) = 5$ being odd.

We thus need to be more clever in our choice of q. By looking again at the Definition 4.2.4, we realize that the right contender for q is simply $q := \frac{\ell+m-k}{2} = 1+t$. This agrees with all our examples, which we gather in Table 4.1, summarizing the behaviour of $(f,g,h)_p$. We also include additional data points, which do not appear in the above example, in our table.

We thus have the following conjecture.

Conjecture 4.3.1. Let f, g, h be three cuspidal new forms of weights k, ℓ, c. Let $t := \frac{\ell+m-k-2}{2}$. We have the following relations:

$$
(f,g,h)_p = (-1)^{t+1}(f,h,g)_p,
$$

i.e. the parity of t determines the symmetry or anti-symmetry of $(f,*,*)_p$.

Remark 12. The difference between our first guess $q = \frac{k+\ell+m}{2}$ and the actual answer $1+t = \frac{\ell+m-k}{2}$, as we will see in Theorem 4.3.2, is simply $q-(1+t) = k$. Thus, the initial erroneous guess $q = \frac{k+\ell+m}{2}$ will only fail when we the first modular form f has odd weight. We will see a similar phenomenon again, in Section 4.3.2, where odd weights will behave differently from even weights.

(k,ℓ,m)	$(k+\ell+m)/2$	$1+t=(\ell+m-k)/2$	Behaviour of $(f,*,*)_p$
(2, 2, 2)	3	1	anti-symmetric
(2, 4, 4)	5	3	anti-symmetric
(2, 6, 6)	7	5	anti-symmetric
(2, 8, 8)	9	7	anti-symmetric
(4, 4, 4)	6	2	symmetric
(4, 4, 6)	7	3	anti-symmetric
(4, 6, 6)	8	4	symmetric
(4, 6, 8)	9	5	anti-symmetric
(4, 8, 8)	10	6	symmetric
(6, 6, 6)	9	3	anti-symmetric
(6, 6, 8)	10	4	symmetric
(6, 8, 8)	11	5	anti-symmetric
(8, 8, 8)	12	4	symmetric
(3, 3, 4)	5	2	symmetric

Table 4.1: Observed behaviour of $(f,g,h)_p$ when switching the order of the second and third variables.

4.3.1.2 Proof

In this section, we rigorously prove that both $(f,*,*)_p$ and $\mathscr{L}_p(f,*,*)$ satisfy symmetry relations, depending precisely on the parity of t.

Theorem 4.3.2. *Let f,g,h be three cuspidal new forms of weights k,ℓ,m. Let $t := \frac{\ell+m-k-2}{2}$. We have the following relations:*

$$\mathscr{L}_p(f,g,h) = (-1)^{t+1}\mathscr{L}_p(f,h,g), \tag{4.12}$$

$$(f,g,h)_p = (-1)^{t+1}(f,h,g)_p, \tag{4.13}$$

i.e. the parity of t determines the symmetry or anti-symmetry of $(f,*,*)_p$ and $\mathscr{L}_p(f,*,*)$.

Proof. We begin the proof by setting up some notation. Let

$$g := \sum_{n\geq 1} a_n(g)q^n; \quad h := \sum_{n\geq 1} a_n(h)q^n;$$
$$G_{-i} := \sum_{p\nmid n} \frac{a_n(g)}{n^i}q^n; \quad H_{-i} := \sum_{p\nmid n} \frac{a_n(h)}{n^i}q^n,$$

so that we have $G_{-i} = \mathrm{d}^{-i}g^{[p]}$ and $H_{-i} = \mathrm{d}^{-i}h^{[p]}$. Consider the sum

$$X := \sum_{i=0}^{t}(-1)^i G_{-1-t+i}H_{-1-i}.$$

We can easily check that

$$\mathrm{d}X = \mathrm{d}\left(\sum_{i=0}^{t}(-1)^i G_{-1-t+i}H_{-1-i}\right) = G_{-1-t}H_0 - (-1)^{t+1}H_{-1-t}G_0.$$

Thus, $G_{-1-t}H_0 - (-1)^{t+1}H_{-1-t}G_0$ is exact and is hence in the kernel of e_{ord}. Indeed, one can readily check that for any form ϕ, $U_{p^{n!}}(\mathrm{d}(\phi))$ is more and more divisible by p as n goes to infinity. So

$$\begin{aligned}0 &= e_{\text{ord}}\left(G_{-1-t}H_0 - (-1)^{t+1}H_{-1-t}G_0\right)\\ &= e_{\text{ord}}\left(\mathrm{d}^{-1-t}\left(g^{[p]}\right)\times h^{[p]} - (-1)^{t+1}g^{[p]}\times \mathrm{d}^{-1-t}\left(h^{[p]}\right)\right) \\ &= e_{\text{ord}}\left(\mathrm{d}^{-1-t}g^{[p]}\times h^{[p]}\right) - (-1)^{t+1}e_{\text{ord}}\left(g^{[p]}\times \mathrm{d}^{-1-t}h^{[p]}\right).\end{aligned} \tag{4.14}$$

By Lemma 2.2.2, $\phi_1^{[p]}\times(V\phi_2)$ is in the kernel of the U_p operator for p-adic modular forms ϕ_1,ϕ_2. Thus, so is $\mathrm{d}^{-1-t}\left(g^{[p]}\right)\times V(U_p(h))$, hence $e_{\text{ord}}\left(\mathrm{d}^{-1-t}\left(g^{[p]}\right)\times V(U_p(h))\right) = 0$. Therefore, as $h^{[p]} = (1-VU_p)h$,

$$\begin{aligned}e_{\text{ord}}\left(\mathrm{d}^{-1-t}\left(g^{[p]}\right)\times h^{[p]}\right) &= e_{\text{ord}}\left(\mathrm{d}^{-1-t}\left(g^{[p]}\right)\times h\right) - e_{\text{ord}}\left(\mathrm{d}^{-1-t}\left(g^{[p]}\right)\times V(U_p(h))\right)\\ &= e_{\text{ord}}\left(\mathrm{d}^{-1-t}\left(g^{[p]}\right)\times h\right).\end{aligned}$$

Similarly, $e_{\text{ord}}\left(g^{[p]}\times \mathrm{d}^{-1-t}\left(h^{[p]}\right)\right) = e_{\text{ord}}\left(g\times \mathrm{d}^{-1-t}\left(h^{[p]}\right)\right)$. Combining this with Equation (4.14), we obtain

$$0 = e_{\text{ord}}\left(\mathrm{d}^{-1-t}g^{[p]}\times h\right) - (-1)^{t+1}e_{\text{ord}}\left(g\times \mathrm{d}^{-1-t}h^{[p]}\right).$$

This implies that $\mathscr{L}_p(f,g,h) = (-1)^{t+1}\mathscr{L}_p(f,h,g)$, proving the first statement of the theorem.

We will now address the second part: Equation (4.13). Let ϖ be the invariant differential associated to $G_{-1-t}H_0 - (-1)^{t+1}H_{-1-t}G_0$. As we have already showed that this is exact, this implies that $\varpi = 0$ is trivial in cohomology, thus

$$\left\langle \omega_f, e_{f^*}(G_{-1-t}H_0 - (-1)^{t+1}H_{-1-t}G_0)\right\rangle = \langle \omega_f, \varpi\rangle = 0,$$

which gives

$$\langle \omega_f, e_{f^*}(G_{-1-t}H_0)\rangle = (-1)^{t+1}\langle \omega_f, e_{f^*}(H_{-1-t}G_0)\rangle. \tag{4.15}$$

As explained above, $U_p(\mathrm{d}^{-1-t}g^{[p]}\times VU_p(h)) = 0$, so $\mathrm{d}^{-1-t}g^{[p]}\times VU_p(h)$ has trivial slope projections and $e_{f^*}(\mathrm{d}^{-1-t}g^{[p]}\times VU_p(h))) = 0$, so

$$e_{f^*}(G_{-1-t}h) = e_{f^*}(G_{-1-t}h - G_{-1-t}\times VU_p(h)) = e_{f^*}(G_{-1-t}H_0).$$

Similarly, $e_{f^*}(H_{-1-t}g) = e_{f^*}(H_{-1-t}G_0)$. Hence, Equation (4.15) gives

$$\langle \omega_f, e_{f^*}(G_{-1-t}h)\rangle = (-1)^{t+1}\langle \omega_f, e_{f^*}(H_{-1-t}g)\rangle.$$

□

Remark 13. One might – rightfully – ask where the term

$$X = \sum_{i=0}^{t}(-1)^i G_{-1-t+i}H_{-1-i},$$

in the proof of Theorem 4.3.2 came from. Here is the heuristic reasoning behind it. We wished to show that $G_{-1-t}H_0 - (-1)^{t+1}H_{-1-t}G_0$ was of the form $\mathrm{d}\mathfrak{X}$ (in order to then say it was in the kernel of e_{ord}). To do that we simply *formally* define

$$\mathfrak{X} := \int G_{-1-t}H_0 - (-1)^{t+1}H_{-1-t}G_0,$$

and *formally* apply integration by parts in hopes of getting a cancellation to the terms involving integrals. The integration by parts formula tells us that

$$\int u^{(x)}v^{(y)} = u^{(x-1)}v^{(y)} - \int u^{(x-1)}v^{(y+1)}, \qquad \forall x, y \in \mathbb{Z}. \tag{4.16}$$

Iteratively repeating Equation (4.16) yields

$$\int u^{(x)}v^{(y)} = \sum_{i=0}^{r}(-1)^i u^{(x-1-i)}v^{(y+i)} + (-1)^{r+1}\int u^{(x-1-r)}v^{(y+1+r)}, \qquad \forall x, y, r \in \mathbb{Z}.$$

Applying this to our case gives us

$$\int H_0 G_{-1-t} = \sum_{i=0}^{r}(-1)^i G_{-1-t+i}H_{-1-i} + (-1)^{r+1}\int G_{-t+r}H_{-1-r}, \qquad \forall r \in \mathbb{Z}. \tag{4.17}$$

Hence, taking $r = t$ in Equation (4.17), we get

$$\begin{aligned}
&\int \big(G_{-1-t}H_0 - (-1)^{t+1}H_{-1-t}G_0\big) \\
&= \int G_{-1-t}H_0 - (-1)^{t+1}\int H_{-1-t}G_0 \\
&= \sum_{i=0}^{t}(-1)^i G_{-1-t+i}H_{-1-i} + (-1)^{t+1}\int G_0 H_{-1-t} - (-1)^{t+1}\int H_{-1-t}G_0 \\
&= \sum_{i=0}^{t}(-1)^i G_{-1-t+i}H_{-1-i}.
\end{aligned}$$

Let us now look back at the expressions for $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m)$ and $(f, g, h)_p$ in terms of projections of p-adic modular forms over the slope 0 and slope $k-1$ subspace. We see, thanks to Equation 4.3 and Definition 4.2.4, that $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m)$ is a multiple of $\ell_{fgh,\alpha}$, while $(f, g, h)_p$ is a linear combination of $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$. Thus, Theorem 4.3.2 is telling us that both quantities $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$ are symmetric or anti-symmetric in the second and third variables – depending on the weights of f, g and h.

Note however that the quantities $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$ are not, individually, fully symmetric in all three variables. To make them symmetric in all three variables, we need to take their linear combination, with the appropriate constants. Such constants are exactly the ones in Definition 4.2.4 of $(f, g, h)_p$, as we will see in the following section.

4.3.2 Cyclic symmetry for $(f, g, h)_p$

We now investigate a much more interesting behaviour : the cyclic symmetry of $(f, g, h)_p$. At the start of the author's investigations, he thought that $(f, g, h)_p$ was perfectly symmetric under any cyclic permutation of the three inputs, i.e. that $(f, g, h)_p = (g, h, f)_p = (h, f, g)_p$. All of the examples computed seemed to agree with this, except for the examples involving odd weights, which were harder to compute and were only done much later, after the examples with even weights. Note as well, that even before computing an example with odd weights, the author already had started suspecting that $(f, g, h)_p$ could not be fully cyclically symmetric (invariant under any cyclic permutation of the three inputs), as this would not agree with Theorem 4.3.2. This will be explained in greater detail in Section 4.3.3, when considering the case of odd weights specifically. Indeed, it turns out that $(f, g, h)_p$ will be perfectly cyclically symmetric precisely when the weights are all even or in the trivial case of when the symbol vanishes.

Unlike Section 4.3.1.1, we do not present our computational evidence here, but rather chose to present it in its own chapter, as the examples involved are longer and more numerous. We now finally present our main theorem, proving symmetry relations for $(f, g, h)_p$, when permuting its inputs.

Theorem 4.3.3. *Let f, g, h be three cuspidal newforms of weights k, ℓ, m. Then $(f, g, h)_p$ satisfies the cyclic symmetry relation*

$$(f, g, h)_p = (-1)^k (g, h, f)_p = (-1)^m (h, f, g)_p.$$

In particular, when the weights are all even, $(f, g, h)_p$ is symmetric when its inputs are cyclically permuted.

Proof. Assume for simplicity that $\chi_f = \chi_g = \chi_h = 1$. We start with the case of weights $k = \ell = m = 2$. In this case, the diagonal cycle $\Delta_{2,2,2}$ is symmetric, as can easily be seen from Definition 3.1 in [DR14]. Recall that $\omega_f \otimes \omega_g \otimes \omega_h$ is given by the Künneth decomposition and is therefore made up of cup products. So by the properties of cup products, we have $\omega_f \otimes \omega_g = -\omega_g \otimes \omega_f$ and $\omega_f \otimes \omega_h = -\omega_h \otimes \omega_f$. We can thus write

$$\mathrm{AJ}_p(\Delta_{2,2,2})(\omega_f \otimes \omega_g \otimes \omega_h) = \mathrm{AJ}_p(\Delta_{2,2,2})(\omega_g \otimes \omega_h \otimes \omega_f).$$

For general weights k, ℓ, m, a variation of the above holds. We will first study the action of permuting the first two coordinates of $(f, g, h)_p$, then the action of permuting the second and third coordinates and finally combine them to obtain the desired result. We make our argument explicit using the functoriality properties of the p-adic Abel Jacobi map. Let

$$r_1 := k - 2, \quad r_2 := \ell - 2, \quad r_3 := m - 2, \quad r := \frac{r_1 + r_2 + r_3}{2},$$

and

$$W := \mathcal{E}^{r_1} \times \mathcal{E}^{r_2} \times \mathcal{E}^{r_3}, \qquad W' := \mathcal{E}^{r_2} \times \mathcal{E}^{r_1} \times \mathcal{E}^{r_3}.$$

Let

$$s : W \longrightarrow W'$$

be the map that permutes the first and second terms. Then s induces permutations on the corresponding Chow groups and De Rham cohomology groups: we have a pushforward s_* on $\mathrm{CH}^{r+2}(W)_0$ and a dual pullback $s^{*,\vee}$ on $\mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W)^\vee$. The functoriality properties of the p-adic Abel Jacobi map with respect to correspondences (see Propositions 1,2 & 4 iii) in [EZZ82]) give us the commuting diagram

$$\begin{array}{ccc} \mathrm{CH}^{r+2}(W)_0 & \xrightarrow{\mathrm{AJ}_p} & \mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W)^\vee \\ \downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle s^{*,\vee}} \\ \mathrm{CH}^{r+2}(W')_0 & \xrightarrow{\mathrm{AJ}_p} & \mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W')^\vee. \end{array}$$

Thus, $\mathrm{AJ}_p s_* = s^{*,\vee}\mathrm{AJ}_p$. Given $Z \in \mathrm{CH}^{r+2}(W)_0$ and some $\omega \in \mathrm{Fil}^{r+2}\mathrm{H}^{2r+3}_{\mathrm{dR}}(W')$, we get

$$\mathrm{AJ}_p(s_*Z)(\omega) = (s^{*,\vee}\mathrm{AJ}_p(Z))(\omega) = \mathrm{AJ}_p(Z)(s^*\omega).$$

We can now apply this to the generalized Gross-Kudla-Schoen diagonal cycle $\Delta_{k,\ell,m}$ and take $\omega = \omega_g \otimes \omega_f \otimes \omega_h$. We see that the action of s^* on ω is given by $s^*(\omega_g \otimes \omega_f \otimes \omega_h) = (-1)^{(k-1)(\ell-1)}(\omega_f \otimes \omega_g \otimes \omega_h)$, by the skew symmetry of cup products (which are part of the Künneth decomposition). Furthermore, the action of s_* on $\Delta_{k,\ell,m}$ is given by $s_*\Delta_{k,\ell,m} = (-1)^{r+(r_1r_2)}\Delta_{\ell,k,m}$. The proof of this is purely combinatorial: one needs to expand Definition 3.3 of $\Delta_{k,\ell,m} \in \mathrm{CH}^{r+2}(W)_0$ in [DR14] and permute two subsets of $\{1, ..., r\}$ of size r_1 and r_2 and intersection of size $r - r_3$. Finally, $r + r_1r_2 + (k-1)(\ell-1) = (k + \ell - m)/2 \mod 2$, therefore we obtain the symmetry formula

$$(f, g, h)_p = (-1)^{(k+\ell-m)/2}(g, f, h)_p.$$

Similarly, $(f, g, h)_p = (-1)^{(\ell+m-k)/2}(f, h, g)_p$. Combining these two symmetry formulas gives

$$(f, g, h)_p = (-1)^k(g, h, f)_p.$$

□

Note that the proof of Theorem 4.3.3 provides an alternative way to prove Equation (4.13) of Theorem 4.3.2. The proof of Theorem 4.3.3 is more conceptual and relies on the functorial properties of the p-adic Abel Jacobi map; whereas the proof of Theorem 4.3.2 is more "hands-on" and is also simpler and more intuitive.

4.3.3 The case of odd weights

Even before establishing Theorem 4.3.3, one can use Theorem 4.3.2 to see that cyclic symmetry for $(f, g, h)_p$ cannot be proven if we allow odd weights. On the one hand we have a conjecture stating that we can cyclically shuffle f, g, h as we wish in $(f, g, h)_p$. On the other hand, we just saw in Theorem 4.3.2 that the order of f, g, h determines whether $(f, g, h)_p$ is symmetric or anti-symmetric in the final two variables.

To be more specific, suppose that we can find modular forms f, g, h of weights k, ℓ, m such that $t_{(k,\ell,m)} := \frac{\ell+m-k-2}{2}$ is odd but $t_{(\ell,m,k)} := \frac{m+k-\ell-2}{2}$ is even. Then we would have by Theorem 4.3.2 that $(f, g, h)_p = (f, h, g)_p$ while $(g, h, f)_p = -(g, f, h)_p$. Hence, if we believe that $(f, g, h)_p$ is invariant under cyclically permuting its three inputs then we'd have

$$(f, g, h)_p = (g, h, f)_p = -(g, f, h)_p = -(f, h, g)_p = -(f, g, h)_p. \tag{4.18}$$

Hence, this would imply that the p-adic triple symbol $(f, g, h)_p$ is vanishing.

Note that this situation cannot happen if all the weights are even. However, if we allow one of the weights to be odd, then we need to have precisely two odd weights and one even weight, as the sum of the three weights must be even. So in the odd weights case, the weights are of the form k, ℓ, m with k even and ℓ, m odd. But then $(\ell + m - k)/2, (\ell + k - m)/2$ and $(k + m - \ell)/2$ cannot all have the same parity. Hence, one of $(f, *, *)_p, (g, *, *)_p, (h, *, *)_p$ must be symmetric and another must be anti-symmetric, in the final two variables. Thus, in the case where the weights are not all even, in order for $(f, g, h)_p$ to satisfy cyclic symmetry, it must be trivial, as described in Equation (4.18).

Experimental evidence shows that this is not actually the case. Hence, when one of the weights is odd, $(f, g, h)_p$ is not always cyclically symmetric. This, of course, is clear from the statement of Theorem 4.3.3, once we know that $(f, g, h)_p$ is not always vanishing. We present the counter-examples in Section 5.3.

4.4 Limitations of $(f, g, h)_p$

The initial hope of the author was to introduce a p-adic L-function, in the usual sense of the term, that would generalize the Garrett-Rankin triple product p-adic L-function, and also satisfy certain symmetry properties. However, it eventually became clear that this was likely to be impossible. Indeed, while we had success with the first two tasks, namely generalizing the Garrett-Rankin triple product p-adic L-function and showing that our new function satisfied symmetry properties, it seems unlikely that this new function would be continuous.

We are not proving here that $(f, g, h)_p$ is actually discontinuous. We are only highlighting the fact that although it is defined as a generalization of a known triple product p-adic L-function, there seems to be no clear reason for it to actually be continuous.

Despite $(f, g, h)_p$ probably not yielding a p-adic L-function, it still is a very interesting object, as it is a symmetrized version of Darmon and Rotger's Garrett-Rankin triple product p-adic L-function. Moreover, our new symbol has a useful application which we discuss in greater detail in Chapter 6. Essentially, it allows us to get a handle on the Poincaré pairing $\langle \omega_f, \phi(\omega_f) \rangle$. This quantity has only so far been computed when f had weight 2. We can now potentially compute it for any f of general integer weight.

The further study of the continuity properties of the p-adic symbol $(f, g, h)_p$ is left for future works. In particular, it would be interesting to study it alongside the triple product p-adic L-function from [AI21], which involves non-zero slope projections.

Chapter 5

Examples

This chapter is dedicated to experimental calculations. Here, we present examples showing how our algorithms from Chapter 3 can be used in practice. We first of all consider modular forms of even weights: overconvergent and nearly overconvergent. Second, we consider modular forms of odd weight with trivial and non-trivial characters.

5.1 Calculations in the overconvergent case

In this section we will concern ourselves with the case where all the modular forms have weight 2. This is the easiest way to ensure that we are working with overconvergent modular forms. In the next section, there will still be instances of overconvergent modular forms coming from classical modular forms which have weight greater than 2.

Example 2. Let us consider the space of new forms $S_2^{\text{new}}(\mathbb{Q}, 57)$ of weight 2 and level 57. Let f, g, h be cuspidal newforms in $S_2^{\text{new}}(\mathbb{Q}, 57)$:

$$\begin{aligned} f &= q - 2q^2 - q^3 + 2q^4 - 3q^5 + 2q^6 - 5q^7 + q^9 + 6q^{10} + q^{11} + ..., \\ g &= q + q^2 + q^3 - q^4 - 2q^5 + q^6 - 3q^8 + q^9 - 2q^{10} + ..., \\ h &= q - 2q^2 + q^3 + 2q^4 + q^5 - 2q^6 + 3q^7 + q^9 - 2q^{10} - 3q^{11} + ... \end{aligned}$$

Let $p := 5$. As in Section 3.2, let $f_{\alpha_{f,p}}$ and $f_{\beta_{f,p}}$ denote the p-stabilizations of f, at some prime p. Then f, g, h are regular and ordinary at p. Using the algorithm described in Section 3.2, we compute the quantities $\ell_{fgh,\alpha}, \ell_{fgh,\beta}, \ell_{ghf,\alpha}, \ell_{ghf,\beta}, \ell_{hfg,\alpha}, \ell_{hfg,\beta}$ and obtain

$$\begin{aligned} \ell_{fgh,\alpha} &= -3774928826965787816511437758179915984738972855613348870149740387513806 \bmod 5^{100} \\ \ell_{fgh,\beta} &= -1600120463087968696799905890349018972704454279824366881678828640068804 \cdot 5^{-1} \bmod 5^{99} \\ \ell_{ghf,\alpha} &= 3414089135682117556340078214096537672013164967359802729338191598002457 \cdot 5 \bmod 5^{101} \\ \ell_{ghf,\beta} &= 319324687965512071716318643272796126647017637487474169128482176479703 \bmod 5^{100}; \\ \ell_{hfg,\alpha} &= 3386642279338565749426053729955310360166771341172640348803607194424548 \cdot 5^{-1} \bmod 5^{99} \\ \ell_{hfg,\beta} &= -1362182692510584292629393424534010351729144263363030199124032659953338 \bmod 5^{100}. \end{aligned}$$

and

$$\ell_{fhg,\alpha} = 3774928826965787816511437758179915984738972855613348870149740387513806 \bmod 5^{100}$$

$$\ell_{fhg,\beta} = 8806793175264059302644094388111179312424900110049379013283342553031 79 \cdot 5^{-1} \bmod 5^{99};$$

$$\ell_{gfh,\alpha} = -341408913568211755634007821409653767201316496735980272933819159800245 7 \cdot 5 \bmod 5^{101}$$

$$\ell_{gfh,\beta} = 1316444872164870993756743549790237920953950571465279247158114744839078 \bmod 5^{100}$$

$$\ell_{hgf,\alpha} = -180892046889654213860259659938973790082035847095459433926304933309642 3 \cdot 5^{-1} \bmod 5^{99}$$

$$\ell_{hgf,\beta} = -1848796736101022160118506527593042717532675210104737039492910699421662 \bmod 5^{100}.$$

Note that we indeed have $\ell_{fgh,\gamma} = -\ell_{fhg,\gamma}$. In order to experimentally verify the symmetry property of Equation (4.5), we will now compute the periods $\Omega_\varphi := \langle \omega_\varphi, \phi(\omega_\varphi) \rangle$ for φ in $\{f, g, h\}$ by using Kedlaya's algorithm (see Section 6.1 for more details on how this is done). We obtain

$$\Omega_f = 295056811991309626265612558389775993563332946790562828653240735140 68 \cdot 5^2 \bmod 5^{100}$$

$$\Omega_g = -159133461381175901704339380528584168392746264473700984619726139435577 \cdot 5 \bmod 5^{100}$$

$$\Omega_h = 784148937089652620613048601058188687937796595870290318348982066196 39 \cdot 5^2 \bmod 5^{100}.$$

Finally, putting everything together we obtain

$$(f, g, h)_p = 587176795250684446515090826597359885828451319074351608232719855765 2 \cdot 5^2 \bmod 5^{100}$$

$$(g, h, f)_p = 942241893372601666712645075776456565816836339229540926165984387920 27 \cdot 5^2 \bmod 5^{100}$$

$$(h, f, g)_p = 328989194731033279961794928605802838532429869011399338836233448557652 \cdot 5^2 \bmod 5^{100}.$$

And we can check that all these values agree modulo 5^{97}.

Example 3. Let $f, g, h \in S_2^{\text{new}}(\mathbb{Q}, 57)$ be as in the previous example, but let $p := 13$. We compute

$$\begin{aligned}
\ell_{fgh,\alpha} &= -1796158008585145947909355232950 05 \bmod 13^{30} \\
\ell_{fgh,\beta} &= -1173874402611247715932653980534105 \cdot 13^{-1} \bmod 13^{29} \\
\ell_{ghf,\alpha} &= 1058442539336085401246122595189804 \bmod 13^{30} \\
\ell_{ghf,\beta} &= 1136250171369817904401024814550290 \bmod 13^{30}; \\
\ell_{hfg,\alpha} &= 634964522103371124972400343164 84 \bmod 13^{30} \\
\ell_{hfg,\beta} &= 8638025999543846308699574365360 7 \cdot 13^{-1} \bmod 13^{29}.
\end{aligned}$$

We also obtain the following periods,

$$\begin{aligned}
\Omega_f &= 747883580536370784038722642576 \cdot 13^2 \bmod 13^{30} \\
\Omega_g &= 6129686138158551610416631571038 2 \cdot 13 \bmod 13^{30} \\
\Omega_h &= 617000202083809365844848126114 9 \cdot 13^2 \bmod 13^{30}.
\end{aligned}$$

Finally, putting everything together we obtain

$$\begin{aligned}
(f, g, h)_p &= 212453319275099778403101936519 8 \cdot 13^2 \bmod 13^{30} \\
(g, h, f)_p &= 212453319275099778403101936519 8 \cdot 13^2 \bmod 13^{30} \\
(h, f, g)_p &= 212453319275099778403101936519 8 \cdot 13^2 \bmod 13^{30}.
\end{aligned}$$

And we can see that all these values agree modulo 13^{30}.

Example 4. Let us consider the space of modular forms of weight 2 and level 37. Let f, g, h be the forms given by:

$$
\begin{aligned}
f &= q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} + ...,\\
g &= q + q^3 - 2q^4 - q^7 - 2q^9 + 3q^{11} + ...,\\
h &= \frac{3}{2} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + 15q^8 + 13q^9 + 18q^{10} + 12q^{11} + ...
\end{aligned}
$$

Note that f and g are cuspidal while h is not. Let $p := 11$. Using the algorithm described in Section 3.2, we compute

$$
\begin{aligned}
\ell_{fgh,\alpha} &= -1244939408173192668968465099397159331106010216807885067920795594379474130716450554533942535066181480 6627 \bmod 11^{100}\\
\ell_{fgh,\beta} &= 3450771149934091507696169875302041808277366926875008578547994224891139307152592068348464449062826700804 \cdot 11 \bmod 5^{101}\\
\ell_{ghf,\alpha} &= 4997848431639863084327118914339170727086527810879217751501458900778446561310884593656247572856403690 8732 \bmod 11^{100}\\
\ell_{ghf,\beta} &= 5286535259177187984025688621942982942790235588885133857599130998385958576260512333331883334133509958 1821 \bmod 11^{100}.
\end{aligned}
$$

We also obtain the following periods,

$$
\begin{aligned}
\Omega_f &= 1560425564171886174125250972396023594348431007710034539825430293654892150571238614470836697519424194957 \cdot 11 \bmod 11^{100}\\
\Omega_g &= 3981102196158132743762690037937916717299478064342121837135751827457318771842317960857493978846502603687 \cdot 11 \bmod 11^{100}.
\end{aligned}
$$

We could not compute Ω_h as our method (based on Kedlaya's algorithm) for computing such periods requires the modular forms to be cuspidal (see Section 6.1). Finally, putting everything together we obtain

$$
\begin{aligned}
(f, g, h)_p &= -4722186621564536539790091975688670502294029447049804730614656886372368926297366120005902994625854 25361 \cdot 11^2 \bmod 11^{101}\\
(g, h, f)_p &= -4722186621564536539790091975688670502294029447049804730614656886372368926297366120005902994625854 25361 \cdot 11^2 \bmod 11^{101}.
\end{aligned}
$$

We then have

$$(f, g, h)_p = (g, h, f)_p \bmod 11^{100}.$$

Example 5. Let now $p := 13$. Then f, g, h are regular and ordinary at p. Using the algorithm described in Section 3.2, we compute $\ell_{fgh,\alpha}, \ell_{fgh,\beta}, \ell_{ghf,\alpha}, \ell_{ghf,\beta}$ and obtain

$$
\begin{aligned}
\ell_{fgh,\alpha} &= 2445454436831732119314778860160512556798031203036093349 5 \bmod 13^{50}\\
\ell_{fgh,\beta} &= -1896539685359418661336041548247409646786063918514792225 7 \bmod 13^{50}\\
\ell_{ghf,\alpha} &= 1128276184382592975263289569512894175796640376095518365 2 \bmod 13^{50}\\
\ell_{ghf,\beta} &= -156154700038499925315841842845669110001639635708239045 38 \cdot 13 \bmod 13^{49}.
\end{aligned}
$$

In order to experimentally verify the symmetry property of equation (4.5), we will now compute the periods $\Omega_\varphi := \langle \omega_\varphi, \phi(\omega_\varphi) \rangle$ for $\varphi \in \{f, g\}$ by using Kedlaya's algorithm. We obtain

$$
\begin{aligned}
\Omega_f &= -7704381609409294131330738919746251252932401437404205 09 \cdot 13 \bmod 13^{50}\\
\Omega_g &= -1402371697069909741021363743582055751673612471659626504 \cdot 13 \bmod 13^{50}.
\end{aligned}
$$

Finally, putting everything together we obtain

$$
\begin{aligned}
(f, g, h)_p &= 4010291763987002512984898210423555182608187109632983 7 \cdot 13^2 \bmod 13^{50}\\
(g, h, f)_p &= -1272648793395700401028689454435438167390335970101952 42 \cdot 13^2 \bmod 13^{50},
\end{aligned}
$$

and we have $(f, g, h)_p = (g, h, f)_p \bmod 13^{50}$.

Example 6. Let us consider the space of new forms $S_2^{\text{new}}(\mathbb{Q}, 99)$ of weight 2 and level 99. Let f, g, h, t be cuspidal newforms:

$$\begin{aligned} f &= q - q^2 - q^4 - 4q^5 - 2q^7 + 3q^8 + 4q^{10} - q^{11} + ..., \\ g &= q + q^2 - q^4 + 4q^5 - 2q^7 - 3q^8 + 4q^{10} + q^{11} + ..., \\ h &= q - q^2 - q^4 + 2q^5 + 4q^7 + 3q^8 - 2q^{10} - q^{11} + ..., \\ t &= q + 2q^2 + 2q^4 - q^5 - 2q^7 - 2q^{10} - q^{11} + ... \end{aligned}$$

Let $p := 5$. Then f, g, h, t are regular and ordinary at p. Using the algorithm described in Section 3.2, we compute $\ell_{fgh,\alpha} = \ell_{fgt,\alpha} = \ell_{gfh,\alpha} = \ell_{gft,\alpha} = \ell_{ght,\alpha} = \ell_{hfg,\alpha} = \ell_{hgt,\alpha} = \ell_{tfg,\alpha} = \ell_{tgh,\alpha} = \ell_{fgh,\beta} = \ell_{fgt,\beta} = \ell_{gfh,\beta} = \ell_{gft,\beta} = \ell_{ght,\beta} = \ell_{hfg,\beta} = \ell_{hgt,\beta} = \ell_{tgf,\beta} = \ell_{tgh,\beta} = 0 \bmod 5^{45}$. And

$$\begin{aligned} \ell_{fht,\alpha} &= -361614522348388352871746267108042222 \cdot 5^{-1} \bmod 5^{49} \\ \ell_{htf,\alpha} &= -208686942810624189369964412110041966 \bmod 5^{50} \\ \ell_{tfh,\alpha} &= 181166991172221855548342645051114 76 \cdot 5^{-1} \bmod 5^{49} \\ \ell_{fht,\beta} &= 106290370512879971595807806478962 94 \bmod 5^{50} \\ \ell_{htf,\beta} &= 123043978937888877798889915728234 77 \bmod 5^{50} \\ \ell_{tfh,\beta} &= -101536758669731495820075539053045 7 \cdot 5^{-2} \bmod 5^{48}. \end{aligned}$$

So we shift our attention to f, h, t. We compute

$$\begin{aligned} (f, h, t)_p &= 2435944747130445855794325238267 89 \cdot 5^2 \bmod 5^{49} \\ (h, t, f)_p &= 9541372104731447714505567914049 14 \cdot 5^2 \bmod 5^{50} \\ (t, f, h)_p &= -1827311667430155259432420367200 86 \cdot 5^2 \bmod 5^{50}. \end{aligned}$$

And we can check that all these values agree modulo 5^{48}.

5.2 Calculations in the non-overconvergent case

In the previous case, when f, g, h all had weight 2, the modular form $\mathrm{d}^{-1}(g^{[p]}) \times h$, that was being projected over f_α and f_β, was overconvergent. This case was addressed in [Lau14]. To make use of the new, more general, algorithms described in Sections 3.1.3 and 3.2, we need to consider modular forms of different weights.

Let f, g, h be modular forms of respective weights k, ℓ, m. We are interested in computing

$$(f, g, h)_p = (-1)^t t! \frac{\langle \omega_f, \phi(\omega_f) \rangle}{p^{k-1}} \left(\frac{\mathcal{E}_1(f)}{\mathcal{E}(f, g, h)} \beta_{f^*} \ell_{fgh,\alpha} + \frac{\tilde{\mathcal{E}}_1(f)}{\tilde{\mathcal{E}}(f, g, h)} \alpha_{f^*} \ell_{fgh,\beta} \right),$$

where $\ell_{fgh,\gamma} := \lambda_{f_\gamma} \left(\mathrm{d}^{-1-t}(g^{[p]}) \times h \right)$ and $t := \frac{\ell+m-k-2}{2}$.

However, we need $0 \leq t \leq \min\{\ell, m\} - 2$ in order to guarantee that $\mathrm{d}^{-1-t}(g^{[p]}) \times h$ is nearly overconvergent. Since the subcase $t = \ell - 2$ ensure the overconvergence of $\mathrm{d}^{-1-t}(g^{[p]}) \times h$, we want to avoid this case in this subsection.

For instance, we could take $(k, \ell, m) = (4, 4, 4)$. Then, $t = 1 \neq \ell - 2$. It is harder to check this result in weight 4. A limitation of our code is that we need the weight

4 to be strictly less than $p-1$. In addition, our method (see Section 6.1) to compute $\Omega_f := \langle \omega_f, \phi(\omega_f) \rangle$, which is based on Kedlaya's algorithm (cf. [Ked01]), only applies when f has weight 2. So we cannot directly check that $(f,g,h)_p = (g,h,f)_p$. We will see however, in this section, how to get around these issues. We can pick a level N so that $S_4^{new}(\mathbb{Q}, N)$ has 4 modular forms f, g, h_1, h_2 and check whether or not we have

$$\frac{(f,g,h_1)_p}{(f,g,h_2)_p} \overset{?}{=} \frac{(g,h_1,f)_p}{(g,h_2,f)_p}. \tag{5.1}$$

Moreover, if we consider the case where a form is repeated twice, as in $f = g$, we then have $\Omega_f = \Omega_g$, and we only need to check that $(g,g,h)_p/\Omega_g = (g,h,g)_p/\Omega_g$. In this case, we can compute both sides of this equality.

Example 7. Take $N := 26$, and $p = 7$. Let $f, g, h \in S_4(\mathbb{Q}, 26)$ be the newforms

$$\begin{aligned}
f &= q + 2q^2 - q^3 + 4q^4 + 17q^5 - 2q^6 - 35q^7 + 8q^8 - 26q^9 + 34q^{10} + 2q^{11} + ...,\\
g &= q + 2q^2 + 4q^3 + 4q^4 - 18q^5 + 8q^6 + 20q^7 + 8q^8 - 11q^9 - 36q^{10} - 48q^{11} + ...,\\
h &= q - 2q^2 + 3q^3 + 4q^4 + 11q^5 - 6q^6 + 19q^7 - 8q^8 - 18q^9 - 22q^{10} - 38q^{11} + ...
\end{aligned}$$

We then would like to experimentally check the symmetry in the three variables, without having to compute the factors $\Omega_f = \langle \omega_f, \phi(\omega_f) \rangle$.

We compute

$$\begin{aligned}
\ell_{ggh,\alpha} &= -853497491248921735765717285309989814027942 \bmod 7^{50}\\
\ell_{ggh,\beta} &= -844796013914105998910733611578948320121855 \bmod 7^{50}\\
\ell_{gfh,\alpha} &= -726086389244708810173847335792542915982534 \bmod 7^{50}\\
\ell_{gfh,\beta} &= -20870274176292581685533778712737818726758 \bmod 7^{50}\\
\ell_{ghg,\alpha} &= 853497491248921735765717285309989814027942 \bmod 7^{50}\\
\ell_{ghg,\beta} &= -662862438250229424238936807802026547626525 \bmod 7^{50}\\
\ell_{hgg,\alpha} &= -569818423040149447329383086481457818344930 \bmod 7^{50}\\
\ell_{hgg,\beta} &= 638471100617667383586834235541410223190670 \bmod 7^{50}\\
\ell_{hgf,\alpha} &= -792212837978234745972056458409084758844364 \bmod 7^{50}\\
\ell_{hgf,\beta} &= -26776706512557383282284171858607065870193 \bmod 7^{50}.
\end{aligned}$$

and obtain

$$\begin{aligned}
(g,g,h)_p/\Omega_g &= -14066462242621113516575344633484539401 \cdot 7^2 \bmod 7^{50},\\
(g,h,f)_p/\Omega_g &= -612101572515327682842831390363209035 9 \bmod 7^{50},\\
(g,h,g)_p/\Omega_g &= -96988334906375203183567201279768747 15 \cdot 7^2 \bmod 7^{50},\\
(h,g,g)_p/\Omega_h &= 13999099609221502029301977732701650652 \cdot 7^2 \bmod 7^{50},\\
(h,g,f)_p/\Omega_h &= 1784957335724921465199755297995800789 \bmod 7^{50}.
\end{aligned}$$

And we can check that,

$$\Omega_g \cdot (g,g,h)_p = \Omega_g \cdot (g,h,g)_p \bmod 7^{43},$$

which means that $(g,g,h)_p = (g,h,g)_p$ modulo a power of 7 potentially slightly smaller that 43 (depending of $\mathrm{val}_7(\Omega_g)$). Similarly,

$$\frac{(g,g,h)_p}{(g,h,f)_p} = \frac{(h,g,g)_p}{(h,g,f)_p} \bmod 7^{43}.$$

Let us now do an example where all the forms are different in the triple product.

Example 8. Take $N := 45$ and let $f, g, h, h_2, h_3 \in S_4(\mathbb{Q}, 45)$ be the cuspidal newforms:

$$\begin{aligned} f &= q - q^2 - 7q^4 - 5q^5 - 24q^7 + 15q^8 + 5q^{10} - 52q^{11}..., \\ g &= q - 3q^2 + q^4 + 5q^5 + 20q^7 + 21q^8 - 15q^{10} + 24q^{11}..., \\ h &= q + 4q^2 + 8q^4 + 5q^5 + 6q^7 + 20q^{10} - 32q^{11} + ..., \\ h_2 &= q + 5q^2 + 17q^4 - 5q^5 - 30q^7 + 45q^8 - 25q^{10} + 50q^{11} + ..., \\ h_3 &= q - 5q^2 + 17q^4 + 5q^5 - 30q^7 - 45q^8 - 25q^{10} - 50q^{11} + ... \end{aligned}$$

Pick $p = 17$, we have $a_{17}(f) \cdot a_{17}(g) \cdot a_{17}(h) \cdot a_{17}(h_2) \cdot a_{17}(h_3) \neq 0$. Consider the p-adic symbols $(\phi_1, \phi_2, \phi_3)_p$, for distinct ϕ_i in $\{f, g, h, h_2, h_3\}$, up to permutations. We have ten potential L-values to compute. Out of these ten, and up to precision 30 (i.e. in $\mathbb{Z}/17^{30}\mathbb{Z}$), seven give us zero. Namely, $\ell_{\phi_1\phi_2\phi_3,\gamma} = 0$ for γ in $\{\alpha, \beta\}$ and (ϕ_1, ϕ_2, ϕ_3) in $\{(f, g, h), (f, g, h_2), (f, h, h_2), (f, h_2, h_3), (g, h, h_3), (g, h_2, h_3), (h, h_2, h_3)\}$. The non-zero values are the ones involving $(\phi_1, \phi_2, \phi_3) \in \{(f, g, h_3), (f, h, h_3), (g, h, h_2)\}$. Now, in order to check Equation 5.1, we compute the following values:

$$\begin{aligned} \ell_{fgh_3,\alpha} &= -452987614719404990529824918211982513 \bmod 17^{30} \\ \ell_{fgh_3,\beta} &= 3024125954105030338283683239626651767 \bmod 17^{30} \\ \ell_{fhh_3,\alpha} &= 2167446936326222112724151737488337903 \bmod 17^{30} \\ \ell_{fhh_3,\beta} &= -2034761566188734529496358243791947123 \bmod 17^{30} \\ \ell_{h_3fg,\alpha} &= -1253203983254546721999333784617671928 \bmod 17^{30} \\ \ell_{h_3fg,\beta} &= 4076342701069946998223745573266518441 \bmod 17^{30} \\ \ell_{h_3fh,\alpha} &= -2522890527148207279455366439012348422 \bmod 17^{30} \\ \ell_{h_3fh,\beta} &= 2741139254319171699307348970094495030 \bmod 17^{30}. \end{aligned}$$

and obtain

$$\begin{aligned} (f, g, h_3)_p/\Omega_f &= -1023342994315815801374020643871 \cdot 17^2 \bmod 17^{30}, \\ (f, h, h_3)_p/\Omega_f &= 68362151699300710278000063432 \cdot 17^2 \bmod 17^{30}, \\ (h_3, f, g)_p/\Omega_{h_3} &= -2631698743570631185431705415466 \cdot 17^2 \bmod 17^{30}, \\ (h_3, f, h)_p/\Omega_{h_3} &= 248547247830740599793540647737 \cdot 17^2 \bmod 17^{30}. \end{aligned}$$

Thus,

$$\frac{(f, g, h_3)_p}{(f, h, h_3)_p} = \frac{(h_3, f, g)_p}{(h_3, f, h)_p} \bmod 17^{25}.$$

We will now consider weights that are not necessarily all the same. We call such cases, case of mixed weights. Fix a level $N \in \mathbb{N}$ and consider cuspidal newforms $f \in S_2(\mathbb{Q}, N)$ and $g, h \in S_4(\mathbb{Q}, N)$. Then, we can consider whether the quantities

$$(f, g, h)_p = (-1)^t t! \frac{\langle \omega_f, \phi(\omega_f) \rangle}{p^{k-1}} \left(\frac{\mathcal{E}_1(f) \beta_{f^*} \lambda_{f^*_\alpha} \left(\mathrm{d}^{-3}(g^{[p]}) \times h \right)}{\mathcal{E}(f, g, h)} + \frac{\tilde{\mathcal{E}}_1(f) \alpha_{f^*} \lambda_{f^*_\beta} \left(\mathrm{d}^{-3}(g^{[p]}) \times h \right)}{\tilde{\mathcal{E}}(f, g, h)} \right),$$

$$(h, f, g)_p = (-1)^t t! \frac{\langle \omega_f, \phi(\omega_f) \rangle}{p^{m-1}} \left(\frac{\mathcal{E}_1(h) \beta_{h^*} \lambda_{h^*_\alpha} \left(\mathrm{d}^{-1}(f^{[p]}) \times g \right)}{\mathcal{E}(h, f, g)} + \frac{\tilde{\mathcal{E}}_1(h) \alpha_{h^*} \lambda_{h^*_\beta} \left(\mathrm{d}^{-1}(f^{[p]}) \times g \right)}{\tilde{\mathcal{E}}(h, f, g)} \right),$$

$$(g,h,f)_p = (-1)^t t! \frac{\langle \omega_f, \phi(\omega_f)\rangle}{p^{\ell-1}} \left(\frac{\mathcal{E}_1(g)\beta_{g^*}\lambda_{g^*_\alpha}\left(\mathrm{d}^{-1}(h^{[p]}) \times f\right)}{\mathcal{E}(g,h,f)} + \frac{\tilde{\mathcal{E}}_1(g)\alpha_{g^*}\lambda_{g^*_\beta}\left(\pi_{\mathrm{oc}}\left(\mathrm{d}^{-1}(h^{[p]}) \times g\right)\right)}{\tilde{\mathcal{E}}(g,h,g)} \right).$$

are equal.

As explained in the beginning of Section 5.2, only the calculation of $(g,h,f)_p$, involves taking a nearly overconvergent projection that is not also an overconvergent projection, since $t_{4,4,2} = (4+2-4-2)/2 = 0 \neq 4-2$. We do this using the methods described in Section 3.1.3.

Example 9. Take $N := 45$, and $p = 17$. Let $f, g, h, h_2, h_3 \in S_4(\mathbb{Q}, 45)$ be the same as in Example 8. Let $f_0 \in S_2(\mathbb{Q}, 45)$ be the newform given by

$$f_0 = q + q^2 - q^4 - q^5 - 3q^8 - q^{10} + 4q^{11} + \ldots.$$

We compute

$$\begin{aligned}
(f_0, f, h_2)_p/\Omega_{f_0} &= 1651322398480093505033606381524 6 \cdot 17^3 \bmod 17^{30},\\
(f, h_2, h_0)_p/\Omega_f &= 1353942137216139610081266472717 7 \cdot 17 \bmod 17^{30},\\
(f_0, h_3, g)_p/\Omega_{f_0} &= -3366884595101012754561302551722 \cdot 17^2 \bmod 17^{30}\\
(h_3, g, f_0)_p/\Omega_{h_3} &= 9339393629152311536018913655 4 \bmod 17^{30}.
\end{aligned}$$

Using Kedlaya's algorithm, we compute

$$\Omega_{f_0} = \langle \omega_{f_0}, \phi(\omega_{f_0})\rangle = 7374052221695942635874395263608 2111 \cdot 17 \bmod 17^{30}.$$

Thus, we deduce that we must have

$$\begin{aligned}
\Omega_f = \Omega_{f_0} \cdot \frac{(f_0, f, h_2)_p/\Omega_{f_0}}{(f, h_2, f_0)_p/\Omega_f} &= -8862546113964214628352195959100 \cdot 17^3 \bmod 17^{27},\\
\Omega_{h_3} = \Omega_{f_0} \cdot \frac{(f_0, h_3, g)_p/\Omega_{f_0}}{(h_3, g, f_0)_p/\Omega_{h_2}} &= -1728830956772474294735820116226 \cdot 17^3 \bmod 17^{26}.
\end{aligned}$$

Similarly for mixed weight $(2,6,6)$, only $(g,h,f)_p$ involves taking a nearly overconvergent projection that is not also an overconvergent projection.

Example 10. Now, take $N = 57$. Let $f \in S_2(\mathbb{Q}, 57)$ and $g, h \in S_6(\mathbb{Q}, 57)$ be the cuspidal newforms given by

$$\begin{aligned}
f &= q - 2q^2 - q^3 + 2q^4 - 3q^5 + 2q^6 - 5q^7 + q^9 + 6q^{10} + q^{11} + \ldots,\\
g &= q - 2q^2 + 9q^3 - 28q^4 - 98q^5 - 18q^6 + 240q^7 + 120q^8 + 81q^9 + 196q^{10} + \ldots,\\
h &= q + 11q^2 + 9q^3 + 89q^4 + 6q^5 + 99q^6 - 176q^7 + 627q^8 + 81q^9 + 66q^{10} + \ldots.
\end{aligned}$$

Pick $p = 11$. We compute

$$\begin{aligned}
(f, g, h)_p/\Omega_f &= -1984158674271658332769712 3 \cdot 11^5 \bmod 11^{30},\\
(g, h, f)_p/\Omega_g &= 4898532676057009152301672 \cdot 11 \bmod 11^{25},\\
(h, f, g)_p/\Omega_h &= 2590652948658337394871975 \cdot 11 \bmod 11^{25}.
\end{aligned}$$

Using Kedlaya's algorithm, we compute

$$\Omega_f = \langle \omega_f, \phi(\omega_f)\rangle = 3530685032509432670092920141 82 \cdot 11 \bmod 11^{30}.$$

Thus, we deduce that we must have

$$\Omega_g = \Omega_f \cdot \frac{(f,g,h)_p/\Omega_f}{(g,h,f)_p)/\Omega_g} = 1935276713161522990409113 \cdot 11^5 \bmod 11^{29},$$
$$\Omega_h = \Omega_f \cdot \frac{(f,g,h)_p/\Omega_f}{(h,f,g)_p/\Omega_h} = 4441442674558133588872252 \cdot 11^5 \bmod 11^{29}.$$

We will now revisit some old examples and see how it is possible to indeed get around the issue of not being able to directly compute the period Ω_ϕ when the weight of the modular form ϕ is not 2.

Example 11. Thanks to Example 9, we know that

$$\Omega_f = -886254611396421462835219595910 0 \cdot 17^3 \bmod 17^{27}$$

and

$$\Omega_{h_3} = -172883095677247429473582011622 6 \cdot 17^3 \bmod 17^{26}.$$

We can thus go back to Example 8 and calculate

$$(f,g,h_3)_p = 997958724864373692770964568 80 \cdot 17^5 \bmod 17^{30},$$
$$(h_3,f,g)_p = 7441712833941157323478381211861 12 \cdot 17^5 \bmod 17^{30},$$
$$(f,h,h_3)_p = 164573667765677253259876978353 \cdot 17^5 \bmod 17^{30},$$
$$(h_3,f,h)_p = -37594455414882431309174268479171 5 \cdot 17^5 \bmod 17^{30}.$$

And we indeed have:

$$(f,g,h_3)_p = (h_3,f,g)_p \bmod 17^{29},$$
$$(f,h,h_3)_p = (h_3,f,h)_p \bmod 17^{29}.$$

We conclude this section with two longer examples involving different modular forms of different weights.

Example 12. Take $N = 21$ and $p = 11$. Let $f_0 \in S_2(\mathbb{Q}, 21)$ and $f, g, h \in S_6(\mathbb{Q}, 21)$ be the cuspidal newforms given by

$$f_0 = q - q^2 + q^3 - q^4 - 2q^5 - q^6 - q^7 + 3q^8 + q^9 + 2q^{10} + 4q^{11} + ...,$$
$$f = q + q^2 - 9q^3 - 31q^4 - 34q^5 - 9q^6 - 49q^7 - 63q^8 + 81q^9 - 34q^{10} - 340q^{11} + ...,$$
$$g = q + 5q^2 + 9q^3 - 7q^4 + 94q^5 + 45q^6 - 49q^7 - 195q^8 + 81q^9 + 470q^{10} + ...,$$
$$h = q + 10q^2 + 9q^3 + 68q^4 - 106q^5 + 90q^6 - 49q^7 + 360q^8 + 81q^9 - 1060q^{10} + ...$$

From Kedlaya's algorithm, we have

$$\Omega_{f_0} = 412797842384875685536202567431940950593928402977097 \cdot 11 \bmod 11^{50}.$$

Consider the triple (f_0, f, g). We can compute

$$(f_0,f,g)_p/\Omega_{f_0} = -2257599454326142239276759004266889152843755460 \cdot 11^5 \bmod 11^{49},$$
$$(f,g,f_0)_p/\Omega_f = -2816145142524823359002534585019971120441513443 \bmod 11^{44},$$
$$(g,f_0,f)_p/\Omega_g = -1202790078682800562850336220378526707376378726 \bmod 11^{44}.$$

This allows us to recover the periods:

$$\begin{aligned}\Omega_f &= \Omega_{f_0} \cdot \frac{(f_0, f, g)_p/\Omega_{f_0}}{(f, g, f_0)_p/\Omega_f} \\ &= -250968918392700398567664486038648683008081751 9 \cdot 11^6 \bmod 11^{50} \\ \Omega_g &= \Omega_{f_0} \cdot \frac{(f_0, f, g)_p/\Omega_{f_0}}{(g, f_0, f)_p/\Omega_g} \\ &= 2597224237884861326788056615405141084095558737 \cdot 11^6 \bmod 11^{50}.\end{aligned} \tag{5.2}$$

Consider now the triple (f_0, f, h). We can compute

$$\begin{aligned}(f_0, f, h)_p/\Omega_{f_0} &= -2847504000645971661684808020815460021295815552 \cdot 11^4 \bmod 11^{50}, \\ (f, h, f_0)_p/\Omega_f &= 2088611347860598644979938539972864115298780 26 \cdot 11^{-1} \bmod 11^{50}, \\ (h, f_0, f)_p/\Omega_h &= 1505623403185356560351173050853572436950394 36 \bmod 11^{50}.\end{aligned}$$

This allows us to recover the periods:

$$\begin{aligned}\Omega_f &= \Omega_{f_0} \cdot \frac{(f_0, f, h)_p/\Omega_{f_0}}{(f, h, f_0)_p/\Omega_f} \\ &= -2659905188070644432593240610595820500448108852 78 \cdot 11^6 \bmod 11^{52} \\ \Omega_h &= \Omega_{f_0} \cdot \frac{(f_0, f, h)_p/\Omega_{f_0}}{(h, f_0, f)_p/\Omega_h} \\ &= 3517323453228488717897255108852367374515726842 66 \cdot 11^5 \bmod 11^{51}.\end{aligned} \tag{5.3}$$

Note that we can also check that the two values we obtained for the period Ω_f from Equations (5.2) and (5.3) math modulo 11^{50}. We can also compute

$$\begin{aligned}(f, g, h)_p/\Omega_f &= -1449471341520532472714863580397344378467971 7 \cdot 11^2 \bmod 11^{44}, \\ (g, h, f)_p/\Omega_g &= 2422280249818398772023030459699296894387061 \cdot 11^2 \bmod 11^{44}, \\ (h, f, g)_p/\Omega_h &= 2930787596521014283263530804024042212003237 \cdot 11^3 \bmod 11^{45}.\end{aligned}$$

This finally allows us to calculate the full values:

$$\begin{aligned}(f, g, h)_p &= 2098691758998671846919428710727628689530731 1 \cdot 11^8 \bmod 11^{50}, \\ (g, h, f)_p &= -2291456031114395478251838824657372595655758 6 \cdot 11^8 \bmod 11^{50}, \\ (h, f, g)_p &= 7861733475215692445486373857156179960213682 \cdot 11^8 \bmod 11^{50}.\end{aligned}$$

And we can check that all these values agree modulo 11^{48}.

Example 13. Take $N = 26$ and $p = 11$. Let $f_0 \in S_2(\mathbb{Q}, 26)$, $f, g, h \in S_4(\mathbb{Q}, 26)$ and $f_1, f_2, f_3 \in S_8(\mathbb{Q}, 26)$ be the cuspidal newforms given by

$$\begin{aligned}f_0 &= q - q^2 + q^3 + q^4 - 3q^5 - q^6 - q^7 - q^8 - 2q^9 + 3q^{10} + 6q^{11} + ..., \\ f_1 &= q + 8q^2 - 27q^3 + 64q^4 - 245q^5 - 216q^6 - 587q^7 + 512q^8 - 1458q^9 + ..., \\ f_2 &= q + 8q^2 - 87q^3 + 64q^4 + 321q^5 - 696q^6 - 181q^7 + 512q^8 + 5382q^9 + ..., \\ f_3 &= q - 8q^2 - 39q^3 + 64q^4 + 385q^5 + 312q^6 - 293q^7 - 512q^8 - 666q^9 + ..., \\ f &= q + 2q^2 - q^3 + 4q^4 + 17q^5 - 2q^6 - 35q^7 + 8q^8 - 26q^9 + 34q^{10} + 2q^{11} + ..., \\ g &= q + 2q^2 + 4q^3 + 4q^4 - 18q^5 + 8q^6 + 20q^7 + 8q^8 - 11q^9 - 36q^{10} - 48q^{11} + ...,\end{aligned}$$

$$h = q - 2q^2 + 3q^3 + 4q^4 + 11q^5 - 6q^6 + 19q^7 - 8q^8 - 18q^9 - 22q^{10} - 38q^{11} + \ldots.$$

From Kedlaya's algorithm, we have

$$\Omega_{f_0} = 390581636402185053366232716528660201295552925543487 \cdot 11 \bmod 11^{50}.$$

Consider the triple (f_0, f_1, f_2). We can compute

$$\begin{aligned}
(f_0, f_1, f_2)_p/\Omega_{f_0} &= -191806241009619865111536935793927995696353 32 \cdot 11^7 \bmod 11^{49}, \\
(f_1, f_2, f_0)_p/\Omega_{f_1} &= 1410920885419217621414191581469345570265606 5 \cdot 11 \bmod 11^{43}, \\
(f_2, f_0, f_1)_p/\Omega_{f_2} &= -779379474878478159925797167495957544635072 6 \cdot 11 \bmod 11^{43}.
\end{aligned}$$

This allows us to recover the periods:

$$\begin{aligned}
\Omega_{f_1} &= \Omega_{f_0} \cdot \frac{(f_0, f_1, f_2)_p/\Omega_{f_0}}{(f_1, f_2, f_0)_p/\Omega_{f_1}} \\
&= 87842792982055783920888690545387643452735 63 \cdot 11^7 \bmod 11^{49} \\
\Omega_{f_2} &= \Omega_{f_0} \cdot \frac{(f_0, f_1, f_2)_p/\Omega_{f_0}}{(f_2, f_0, f_1)_p/\Omega_{f_2}} \\
&= -14996446534128706542282744967596509831174973 \cdot 11^7 \bmod 11^{49}.
\end{aligned}$$

Now in order to recover $\Omega_f, \Omega_g, \Omega_h$, we compute

$$\begin{aligned}
(f, f_1, f_3)_p/\Omega_f &= 10754233019386849803882649112958846498341 12 \cdot 11^6 \bmod 11^{47}, \\
(f_1, f_3, f)_p/\Omega_{f_1} &= -13716503028636482837493563350397024875730 85 \cdot 11^2 \bmod 11^{43}, \\
(g, f_1, f_3)_p/\Omega_g &= 13661483455838683030726573564843649455451 96 \cdot 11^5 \bmod 11^{46} \\
(f_1, f_3, g)_p/\Omega_{f_1} &= 14582242522544761160402094298499885974070 90 \cdot 11^2 \bmod 11^{43}, \\
(h, f_2, f_2)_p/\Omega_h &= -22538595761447167385985178477156106689979 56 \cdot 11^6 \bmod 11^{47} \\
(f_2, f_2, h)_p/\Omega_{f_2} &= -11794537719455345117158672128692719330993 33 \cdot 11^2 \bmod 11^{43}.
\end{aligned}$$

This allows us to recover the periods:

$$\begin{aligned}
\Omega_f &= \Omega_{f_1} \cdot \frac{(f_1, f_3, f)_p/\Omega_{f_1}}{(f, f_1, f_3)_p/\Omega_f} \\
&= -89977488745000891823159385117660744807295 8 \cdot 11^3 \bmod 11^{44} \\
\Omega_g &= \Omega_{f_1} \cdot \frac{(f_1, f_3, g)_p/\Omega_{f_1}}{(g, f_1, f_3)_p/\Omega_g} \\
&= 36578899966340566317653585313947952362533 \cdot 11^4 \bmod 11^{45} \\
\Omega_h &= \Omega_{f_2} \cdot \frac{(f_2, f_2, h)_p/\Omega_{f_2}}{(h, f_2, f_2)_p/\Omega_h} \\
&= -17789563642955619254879952723617149702193 39 \cdot 11^3 \bmod 11^{44}.
\end{aligned}$$

We finally can calculate the full values:

$$\begin{aligned}
(f, g, h)_p &= 47935916785738964877959347835339957789102 0 \cdot 11^5 \bmod 11^{46}, \\
(g, h, f)_p &= 13995060165988180904530465018727915146345 46 \cdot 11^5 \bmod 11^{46}, \\
(h, f, g)_p &= 20952268046716054487915109830703805399772 12 \cdot 11^5 \bmod 11^{46}.
\end{aligned}$$

And we can check that all these values agree modulo 11^{43}.

By looking at all the examples given in this section, we can observe that the p-adic valuation of the period Ω_f grows with the weight of f. It would be interesting to study these periods further. Moreover, thanks to the reciprocity result given in Theorem 4.3.3, one could try to assign a value to a non-cuspidal modular form h by considering a balanced triple (f, g, h) where f and g are cuspidal. We leave this for future works.

5.3 Failure of symmetry for odd weights

We present here the examples that we alluded to at the end of Section 4.3.3. They show that we cannot have perfect symmetry when the weights are odd, as this would imply that our p-adic symbol is vanishing – which is not consistent with our experimental computations below.

Example 14. Let χ be the Legendre symbol $\left(\frac{\cdot}{11}\right)$. Let $f_0 \in S_2(\mathbb{Q}, \Gamma_0(11))$ and $f \in S_7(\mathbb{Q}, \Gamma_1(11), \chi)$ be the cuspidal newforms given by

$$\begin{aligned} f_0 &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} + ..., \\ f &= q + 10q^3 + 64q^4 + 74q^5 - 629q^9 - 1331q^{11} + ... \end{aligned}$$

Pick $p = 23$. We have $a_p(f_0), a_p(f) \neq 0$. From Kedlaya's algorithm, we have

$$\Omega_{f_0} = 1908316926377665890962138787495804830512022265904787726295870685765 \cdot 23 \bmod 23^{50}.$$

Using the algorithms described in Sections 3.1 and 3.2 we compute

$$(f_0, f, f)_p/\Omega_{f_0} = 1280035183781782805368449759120961228047405761733504080 3146 \cdot 23^6 \bmod 23^{49}.$$

We thus can calculate the full values:

$$(f_0, f, f)_p = -4287211555949028297914812212960436193845556190314173166613 \cdot 23^7 \bmod 23^{50}.$$

In particular, $(f_0, f, f)_p \neq 0$.

Example 15. Let χ be the Legendre symbol $\left(\frac{\cdot}{11}\right)$. Let $f_0 \in S_2(\mathbb{Q}, \Gamma_0(11))$ and $f \in S_5(\mathbb{Q}, \Gamma_1(11), \chi)$ be the cuspidal newforms given by

$$\begin{aligned} f_0 &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} + ..., \\ f &= q + 7q^3 + 16q^4 - 49q^5 - 32q^9 + 121q^{11} + ... \end{aligned}$$

Pick $p = 23$. We have $a_p(f_0), a_p(f) \neq 0$. From Kedlaya's algorithm, we have

$$\Omega_{f_0} = 756130671642371484124056479062727033371 \cdot 23 \bmod 23^{30}.$$

Using the algorithms described in Sections 3.1 and 3.2 we compute

$$(f_0, f, f)_p/\Omega_{f_0} = 1009184263622171764784067077357 4570 \cdot 23^4 \bmod 23^{30}.$$

We thus can calculate the full values:

$$(f_0, f, f)_p = 1019390402799206113791426684677 46527 \cdot 23^5 \bmod 23^{30}.$$

In particular, $(f_0, f, f)_p \neq 0$.

We now present a examples that doesn't involve any characters.

Example 16. Let $f \in S_2(\mathbb{Q}, \Gamma_0(15))$ and $g, h \in S_3(\mathbb{Q}, \Gamma_1(15))$ be the cuspidal newforms given by

$$\begin{aligned} f &= q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + q^9 - q^{10} - 4q^{11} + ..., \\ g &= q + q^2 - 3q^3 - 3q^4 + 5q^5 - 3q^6 - 7q^8 + 9q^9 + 5q^{10} + ..., \\ h &= q - q^2 + 3q^3 - 3q^4 - 5q^5 - 3q^6 + 7q^8 + 9q^9 + 5q^{10} + ... \end{aligned}$$

Pick $p = 13$. Note that we actually have $a_p(f) \neq 0$ but $a_p(g) = a_p(h) = 0$ here. This doesn't pose any issues to our algorithms. From Kedlaya's algorithm, we have

$$\Omega_f = 75322919821981880121771213979989222236741330896142626 8 \cdot 13 \bmod 13^{50}.$$

Using the algorithms described in Sections 3.1 and 3.2 we compute

$$\begin{aligned} (f, g, h)_p/\Omega_f &= -2518718285900663610678290074698083293592484983936679 \bmod 13^{47}. \\ (f, g, h)_p/\Omega_f &= -2518718285900663610678290074698083293592484983936679 \bmod 13^{47}. \end{aligned}$$

We thus can calculate the full values:

$$(f, g, h)_p = (f, g, h)_p = -1035139817698281500403361859276752699075519329405750 2 \cdot 13 \bmod 13^{48}.$$

In particular, $(f, g, h)_p \neq 0$ and is symmetric (in the 2nd and 3rd variables), which is consistent with the fact that $t_{2,3,3} = 1$ is odd.

Example 17. Let $f \in S_2(\mathbb{Q}, \Gamma_0(15))$ and $g, h \in S_5(\mathbb{Q}, \Gamma_1(15))$ be the cuspidal newforms given by

$$\begin{aligned} f &= q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + q^9 - q^{10} - 4q^{11} + ..., \\ g &= q + 7q^2 - 9q^3 + 33q^4 - 25q^5 - 63q^6 + 119q^8 + 81q^9 - 175q^{10} + ..., \\ h &= q - 7q^2 + 9q^3 + 33q^4 + 25q^5 - 63q^6 - 119q^8 + 81q^9 - 175q^{10} + ... \end{aligned}$$

Pick $p = 17$. We have $a_p(f), a_p(g), a_p(h) \neq 0$. From Kedlaya's algorithm, we have

$$\Omega_f = 2184308939957489165307930804130219760421038240714675194148 76 \cdot 17 \bmod 17^{50}.$$

Using the algorithms described in Sections 3.1 and 3.2 we compute

$$\begin{aligned} (f, g, h)_p/\Omega_f &= 1162038638235844836824541386461967371597258751784343531 8 \cdot 17^4 \bmod 17^{49}. \\ (f, h, g)_p/\Omega_f &= 1162038638235844836824541386461967371597258751784343531 8 \cdot 17^4 \bmod 17^{49}. \end{aligned}$$

We thus can calculate the full values:

$$(f, g, h)_p = (f, g, h)_p = 8960308425349268584612725752076582316781113083897858380 \cdot 17^5 \bmod 17^{50}.$$

In particular, $(f, g, h)_p \neq 0$ and is symmetric (in the 2nd and 3rd variables), which is consistent with the fact that $t_{2,3,3} = 3$ is odd.

Chapter 6

Computing Poincaré pairings

The Poincaré pairing described in Section 2.3 plays an important role in number theory and appears in many different areas of this field (see [DL21], [DLR16] and Section III.5 of [Nik11] for instance). It is somewhat mysterious as although there are many ways of describing it theoretically, we cannot efficiently compute it algorithmically.

The main reason why we are specifically interested in the Poincaré pairing is that it appears in Definition 4.2.4 of our p-adic triple symbol $(f, g, h)_p$, which is at the heart of this thesis. Moreover, in order to demonstrate that our implementation of our new algorithms from Chapter 3 functions properly, we would like to compute $(f, g, h)_p, (g, h, f)_p$ and $(h, f, g)_p$, and check that these quantities are all equal (up to a sign, in the case of odd weight, as in Theorem 4.3.3).

Using the algorithms of Sections 3.1.3 and 3.2, we can compute the quantities $\ell_{fgh,\alpha}$ and $\ell_{fgh,\beta}$ appearing in Definition 4.2.4. Furthermore, $\mathcal{E}_1(f), \mathcal{E}(f, g, h), \tilde{\mathcal{E}}_1(f), \tilde{\mathcal{E}}(f, g, h)$ all have a closed form given in Equation (4.1). Lastly, $\alpha_{f^*}, \beta_{f^*}$ can be easily computed as roots of a Hecke polynomial (see Equation 3.14). Thus the only remaining factor in Equation (4.10) that is non-trivial to calculate is the period $\Omega_f := \langle \omega_f, \phi(\omega_f) \rangle$.

In Section 5.2, we have explained how one can get around the issue of computing the period Ω_f and still show that our new algorithms from Chapter 3 work properly. We have thus so far avoided the need to explicitly compute the Poincaré pairing $\langle \omega_f, \phi(\omega_f) \rangle$. In this Chapter however, we go back to the question of computing Poincaré pairings and try to address it using a new approach.

In Section 2.3, we saw that there were no known ways to efficiently compute a general Poincaré pairing $\langle \omega, \eta \rangle$. But we are actually only interested in Poincaré pairings of the form $\langle \omega_f, \phi(\omega_f) \rangle$, which do appear in the literature and are not solely restricted to our p-adic triple symbol formulas. This makes the problem of computing these pairings slightly more contained.

6.1 The case of weight 2

In the case where f is a newform (with rational coefficients) of weight 2, the above problem has already been considered in Section 4 of [DL21], where the authors used the following trick to easily calculate Ω_f. The method relies on Kedlaya's algorithm (cf. [Ked01]).

In general, given an elliptic curve E, of conductor N, over $\mathbb{Q}$, the modularity theorem [Wil95, TW95, BCDT01] proves the existence of a surjective map $\pi : X_0(N) \longrightarrow E$ defined over $\mathbb{Q}$. There exists a unique such map of minimal degree, up to composing with automorphisms of E. We call the degree of this map the modular degree of E and denote it by m_E. The modular degree can be computed using MAGMA [BCP97].

Let now E be the elliptic curve associated to f. The differential $\omega_f = \sum_n a_n(f)q^n \frac{\mathrm{d}q}{q}$ corresponds to the invariant differential $\omega_E := \frac{\mathrm{d}x}{y}$ of the elliptic curve E. Computing the Poincaré pairing $\langle \omega_f, \phi(\omega_f)\rangle$ now amounts to calculating $\langle \omega_E, \mathrm{Frob}(\omega_E)\rangle$, up to including the modular degree m_E of E:

$$\langle \omega_f, \phi(\omega_f)\rangle = m_E \left\langle \omega_E, \mathrm{Frob}(\omega_E)\right\rangle . \tag{6.1}$$

The reason behind this is that the correspondence between ω_f and ω_E is not perfect, and the modular degree m_E of E is needed as a correction factor.

Let $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ be the matrix representing the action of Frobenius, up to precision p^m, on the differentials $\omega_E = \frac{\mathrm{d}x}{y}$ and $\eta_E := x\frac{\mathrm{d}x}{y}$. Then,

$$\langle \omega_E, M\omega_E\rangle = \langle \omega_E, M_{11}\omega_E + M_{21}\eta_E\rangle = M_{21}.$$

Hence, the period Ω_f is simply given by

$$\Omega_f = m_E M_{21} \mod p^m.$$

Finally, the matrix M can be efficiently computed via Kedlaya's algorithm. This hence gives us an efficient way to compute the pairing $\langle \omega_f, \phi(\omega_f)\rangle$ when f is a newform of weight 2 with rational coefficients.

6.2 The case of general weights

In the case where f has weight strictly greater than 2, we cannot use the above trick anymore. Moreover, calculating a Poincaré pairing directly by using the definition given in Equation (2.16) does not seem feasible in our case. We will thus avoid this direct approach and instead we resort to a workaround by exploiting the symmetry of $(f, g, h)_p$.

Indeed, given a modular form f of weight k and level N, we will exploit the symmetry discussed in Theorem 4.3.3 together with the method described in Section 6.1 allowing us to compute pairings $\langle \omega_\varphi, \phi(\omega_\varphi)\rangle$ for φ of weight 2.

Start by picking an eigenform φ with rational coefficients of weight 2 and level N. Let g be any modular form of weight ℓ and level N such that the triple of modular forms (f, g, φ) is balanced. Note that if k is greater than 2, then one needs to have $\ell = k$ in order for (f, g, φ) to be balanced. Note that there is some freedom in the choice of φ and g, and that in most cases, there will be many valid options for φ and g. In particular, one can take for example $g := f$. Next, we compute the quantities:

$$(\varphi, f, g)_p, \qquad (f, g, \varphi)_p/\Omega_f.$$

Note that computing $(\varphi, f, g)_p$ involves computing Ω_φ, which is can be done by simply following the method described in Section 6.1, as φ has weight 2. Moreover, computing

$(f,g,\varphi)_p/\Omega_f$ doesn't involve any Poincaré pairings, as we have divided out by the period Ω_f. Finally, we obtain:

$$\Omega_f = \frac{(\varphi, f, g)_p}{(f,g,\varphi)_p/\Omega_f}.$$

The method described above will not always work on the first try, as the p-adic triple symbol $(f,g,\varphi)_p$ might happen to vanish. But one can hope that by trying all the valid combinations of φ and g, one will be able to recover the period Ω_f.

In the case where the above method doesn't work, one can still proceed further, as follows, in order to compute the Poincaré pairing $\Omega_f := \langle \omega_f, \phi(\omega_f)\rangle \mod p^m$, for a modular form f of weight $k > 2$, level N, and a precision level $m \in \mathbb{N}$.

(1) Start by picking a new modular form f_0 of weight $k_0 > 2$ and level N. Then, we apply the above method to obtain Ω_{f_0}.

If this step fails, we start over with a different form f_0. Now that we have a modular form f_0 of weight k_0 and known period Ω_{f_0}, we can proceed in a similar way to the method described earlier in Section 6.2 to recover Ω_f. In the following, the form f_0 will play the role that φ played above.

(2) Pick a modular form g of any weight ℓ such that the triple (f, f_0, g) is balanced, and check if $(f,g,f_0)_p/\Omega_f$ is non-zero (modulo p^m).

If $(f,g,f_0)_p/\Omega_f = 0 \mod p^m$, then we can pick a new form g of weight ℓ, ensuring (f, f_0, g) remains balanced, and repeat Step 2. Since $k_0 \neq 2$, we have a much greater freedom in picking g while still ensuring that the triple (f, f_0, g) is balanced. If we still are unlucky after a few tries (if $(f,g,f_0)_p/\Omega_f$ keeps vanishing), we can decide to go back to Step 1, pick a new form f_0 and start over from there.

(3) Compute $(f_0, f, g)_p$ and return

$$\frac{(f_0, f, g)_p}{(f,g,f_0)_p/\Omega_f} = \Omega_f \mod p^m. \tag{6.2}$$

Remark 14. In the case where both numerator and denominator, on the left hand side of Equation (6.2), are divisible by a power of p, there might be a slight loss of precision. However, one can easily keep track of that in practice.

We note that a particular feature of this algorithm is that there does not seem to be any obstruction for it working with non cuspidal modular forms. Finally, for concrete and detailed examples of the use of this algorithm, see Examples 9, 10, 12 and 13 in Section 5.2.

Chapter 7

The challenges and uses of experimental algorithms

In this chapter, we will discuss some aspects at the interface of experimental computations and theoretical research in number theory. We will particularly describe how computations can often help lead and correct theoretical discoveries.

Such matters are not conventionally included in a thesis – and even less in a research paper. However, as the nature of the author's work is heavily algorithmic and computational, it makes sense to comment on and highlight certain aspects of his experimental research, especially the parts which had a noticeable impact on the examples presented in this work.

Having gone through most of this thesis by now, the reader has the advantage of being presented with a polished and coherent version of the theory as well as a selection of supporting examples. However, this clear presentation hides the many mistakes that have occurred throughout the elaboration of this thesis. We will present here an error that occurred in the theoretical calculations done by the author, and how it was caught by experimental calculations, thus highlighting the importance of experimental computations.

Looking back at the discussion surrounding Equation (4.9), we put ourselves back in the context where the author was seeking the appropriate multiple of $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ to ensure the resulting quantity was symmetric[1]. To find the right *coefficients*, the plan was to simply compute various multiples of $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ and see which ones were indeed symmetric. Let us call this unknown coefficient $\iota = \iota_{k,\ell,m}$, where k, ℓ, m are the weights of f, g, h respectively. The author suspected that $\iota = \iota_{k,\ell,m}$ was one of the terms appearing in $\frac{(-1)^t}{t!} \frac{\mathcal{E}(f,g,h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)}$, so there were only finitely many contenders for ι to test.

The problem with this initial approach is that it was not possible to directly compute $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ when the weight of f was greater than 2. Indeed, the formula given by Theorem 4.2.3, involved the calculation of the period $\Omega_f := \langle \omega_f, \phi(\omega_f) \rangle$, which can only be done *by direct methods* in weight 2 (as explained in Chapter 6).

[1] Keep in mind that this work was done much before the development of the proof of Theorem 4.3.3. Indeed, Theorem 4.3.3 came after the accumulation of experimental evidence suggesting its veracity.

Nonetheless, the weight 2 case did reveal valuable information as it allowed the author to see that $\iota_{2,2,2} = 1$, ruling out the factors $\mathcal{E}_0(f), \mathcal{E}_1(f), \mathcal{E}(f, g, h)$. The author was thus left with only one factor to check: $(-1)^t/t!$. At this stage, the author decided to ignore the factor $(-1)^t$, as in the case of even weights, it would no make any difference, and in the case of odd weight, the author was starting to suspect that perfect symmetry did not hold, so the issue was moot[2]. The author was now trying to address the potential factor $1/t!$, in the even weight case.

The problem is that this factor cancels out if all the forms have the same weights, so experimental evidence involving $k = \ell = m$ would not be conclusive. So, one would need to take for example $(k, \ell, m) = (4, 6, 8)$ in order to check whether the factor $1/t!$ is needed for the symmetry relation to hold. However, remember that we can only compute Ω_ϕ if ϕ has weight 2. So all our computations so far (many of which involved different weights that were not just 2) did not indicate the need to add $1/t!$ because this potential factor was merged with the unknown period Ω_ϕ. To truly see that we need to add $1/t!$, the author needed to conduct an experiment involving many modular forms, of weights $(2, w_1, w_1)$, $(2, w_2, w_2)$ and (w_1, w_2, w_2), for $w_1 \neq w_2$, as is done below in Examples 18 and 19.

In order to establish whether or not the factor $1/t!$ is needed in Definition 4.2.4 of $(f, g, h)_p$, we need to compute an extensive example involving many modular forms. We set up the following notation

$$\mathrm{AJ}_p^\circ(\Delta)(\omega_{\phi_1} \otimes \omega_{\phi_2} \otimes \omega_{\phi_3}) := \mathrm{AJ}_p(\Delta)(\omega_{\phi_1} \otimes \omega_{\phi_2} \otimes \omega_{\phi_3})/\Omega_{\phi_1}.$$

We are essentially trying to find out if the constants $\iota_{k,\ell,m}, \iota_{\ell,m,k}, \iota_{m,k,\ell}$ are 1 or not, in order for

$$\frac{\Omega_f}{\iota_{k,\ell,m}}\mathrm{AJ}_p^\circ(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h) = \frac{\Omega_g}{\iota_{\ell,m,k}}\mathrm{AJ}_p^\circ(\Delta)(\omega_g \otimes \omega_h \otimes \omega_f) = \frac{\Omega_h}{\iota_{m,k,\ell}}\mathrm{AJ}_p^\circ(\Delta)(\omega_h \otimes \omega_f \otimes \omega_g) \quad (7.1)$$

to hold. And while we cannot compute any of $\iota_{k,\ell,m}, \iota_{\ell,m,k}, \iota_{m,k,\ell}$ individually, we can nonetheless compute some of their ratios, as in Equations (7.4) and (7.5), which will be enough for our purposes. The examples below make this clear.

Before presenting the computational examples that allowed the author to discover ι, we pause to reflect on what we expect to happen. Having read Chapter 4, and being aware of Theorem 4.3.3, we know that in the set up of Equation (7.1), the factors $\iota_{k,\ell,m}, \iota_{\ell,m,k}, \iota_{m,k,\ell}$ all must be 1. Indeed, they all should be 1 in the absence of any mistakes. The examples below tell the story of how the author, while looking for a potential missing factor ι, realized that no missing factor was needed, but discovered instead that all his formulas were off by a factor of $t!$ due to a typo that was made at the start of his investigations on his thesis research topic. Indeed, the author had mistakenly forgot the factor $t!$ both in Lemma 4.2.2 and Theorem 4.2.3. This mistaken was likely due to the fact that the author initially started working with forms of weight 2 (where $t = 2$) and when he moved onto higher weight forms, he carried over some of his formulas without modifying them appropriately.

[2]In hindsight, we now know that there still is a symmetry relation in the odd weight case, as described in Theorem 4.3.3, just not the perfect symmetry relation that the author was initially looking for.

Example 18. Take $N = 45$ and $p = 11$. Let $f_0 \in S_2(\mathbb{Q}, 45)$, $f_1, f_4 \in S_4(\mathbb{Q}, 45)$, $g_1 \in S_6(\mathbb{Q}, 45)$ and $h_2, h_3, h_5, h_7 \in S_8(\mathbb{Q}, 45)$ be the cuspidal newforms given by

$$\begin{aligned}
f_0 &= q + q^2 - q^4 - q^5 - 3q^8 - q^{10} + 4q^{11} + ...,\\
f_1 &= q - q^2 - 7q^4 - 5q^5 - 24q^7 + 15q^8 + 5q^{10} - 52q^{11} + ...,\\
f_4 &= q + 5q^2 + 17q^4 - 5q^5 - 30q^7 + 45q^8 - 25q^{10} + 50q^{11} + ...,\\
g_1 &= q + 2q^2 - 28q^4 + 25q^5 - 132q^7 - 120q^8 + 50q^{10} - 472q^{11} + ...,\\
h_2 &= q - 5q^2 - 103q^4 + 125q^5 + 930q^7 + 1155q^8 - 625q^{10} - 8450q^{11} + ...,\\
h_3 &= q + 10q^2 - 28q^4 + 125q^5 - 1170q^7 - 1560q^8 + 1250q^{10} + 2650q^{11} + ...,\\
h_5 &= q + 13q^2 + 41q^4 + 125q^5 + 1380q^7 - 1131q^8 + 1625q^{10} + 3304q^{11} + ...,\\
h_7 &= q + 22q^2 + 356q^4 + 125q^5 - 420q^7 + 5016q^8 + 2750q^{10} + 2944q^{11} + ...
\end{aligned}$$

From Kedlaya's algorithm, we have

$$\Omega_{f_0} = 7517972785661700900100000680995759475024829176945 1 \cdot 11 \bmod 11^{50}.$$

Consider the triples (f_0, f_1, f_4), (f_0, h_3, h_5), (f_0, h_3, h_7) and (f_0, h_2, h_5). We can compute

$$\begin{aligned}
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{f_1} \otimes \omega_{f_4}) &= 45388181573533709018757213276839454745620940075 \cdot 11^3 \bmod 11^{50},\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_1} \otimes \omega_{f_4} \otimes \omega_{f_0}) &= -212898038418473542916671372687304735742078143 05 \cdot 11 \bmod 11^{50}\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_3} \otimes \omega_{h_5}) &= -78837445849786093760246356704844333636618 36 \cdot 11^7 \bmod 11^{50},\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{h_5} \otimes \omega_{f_0}) &= -59714834898782653752000193761748707070057 90 \cdot 11 \bmod 11^{50}\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_3} \otimes \omega_{h_7}) &= -15633881059909298982283096801121556811369 58 \cdot 11^6 \bmod 11^{48},\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{h_7} \otimes \omega_{f_0}) &= 95160963247860872569682136301791456091753 0 \bmod 11^{48}\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_2} \otimes \omega_{h_5}) &= 61448471154158416611151843736283831610740 35 \cdot 11^7 \bmod 11^{50},\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{h_5} \otimes \omega_{f_0}) &= 23909428463197686164468906797722706561621936 \cdot 11 \bmod 11^{50}.
\end{aligned}$$

This allows us to recover the periods (up to some ratio of ι's):

$$\begin{aligned}
\Omega_{f_1} \cdot \frac{\iota_{2,4,4}}{\iota_{4,4,2}} &= \Omega_{f_0} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{f_1} \otimes \omega_{f_4})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_1} \otimes \omega_{f_4} \otimes \omega_{f_0})}\\
&= -23608675761473250885248131950403626809533191786 5 \cdot 11^2 \bmod 11^{48}\\
\Omega_{h_3} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}} &= \Omega_{f_0} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_3} \otimes \omega_{h_5})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{h_5} \otimes \omega_{f_0})} \qquad (7.2)\\
&= 80853523941033724922091197581646624910310 19 \cdot 11^6 \bmod 11^{48}\\
\Omega_{h_3} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}} &= \Omega_{f_0} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_3} \otimes \omega_{h_7})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{h_7} \otimes \omega_{f_0})} \qquad (7.3)\\
&= 15807370690572584268833685065377414422627 60 \cdot 11^6 \bmod 11^{47}\\
\Omega_{h_2} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}} &= \Omega_{f_0} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_2} \otimes \omega_{h_5})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{h_5} \otimes \omega_{f_0})}\\
&= -13128944487668525963364745889232157275775528 \cdot 11^6 \bmod 11^{48}.
\end{aligned}$$

Note that the values of Ω_{h_3} at (7.2) and (7.3) agree modulo 11^{44}.

Consider now the triples (f_1, g_1, h_2) and (f_1, g_1, h_3). We can compute

$$\begin{aligned}
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_1} \otimes \omega_{g_1} \otimes \omega_{h_2}) &= -16385429971934052493894348074276020562519981 \cdot 11^5 \bmod 11^{50},\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{f_1} \otimes \omega_{g_1}) &= 20169725681001505412372580117718650758563247 \cdot 11 \bmod 11^{50}\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_1} \otimes \omega_{g_1} \otimes \omega_{h_3}) &= -18654060703224527449855303334476367028551545 \cdot 11^5 \bmod 11^{50},\\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{f_1} \otimes \omega_{g_1}) &= -21420916386748394556607704598255082490950756 \cdot 11 \bmod 11^{50}.
\end{aligned}$$

Now, Equation (7.1) allow us to write:

$$\frac{\iota_{4,6,8}}{\iota_{8,4,6}} = \frac{\Omega_{f_1}}{\Omega_{h_2}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_1} \otimes \omega_{g_1} \otimes \omega_{h_2})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{f_1} \otimes \omega_{g_1})}$$
$$\frac{\iota_{4,6,8}}{\iota_{8,4,6}} = \frac{\Omega_{f_1}}{\Omega_{h_3}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_1} \otimes \omega_{g_1} \otimes \omega_{h_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{f_1} \otimes \omega_{g_1})}.$$

We thus finally obtain:

$$\begin{aligned} \frac{\iota_{4,6,8}}{\iota_{8,4,6}} \cdot \frac{\iota_{2,4,4}}{\iota_{4,4,2}} \cdot \frac{\iota_{8,8,2}}{\iota_{2,8,8}} &= \frac{\Omega_{f_1} \cdot \frac{\iota_{2,4,4}}{\iota_{4,4,2}}}{\Omega_{h_2} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_1} \otimes \omega_{g_1} \otimes \omega_{h_2})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{f_1} \otimes \omega_{g_1})} \\ &= 15 \bmod 11^{38} \\ \frac{\iota_{4,6,8}}{\iota_{8,4,6}} \cdot \frac{\iota_{2,4,4}}{\iota_{4,4,2}} \cdot \frac{\iota_{8,8,2}}{\iota_{2,8,8}} &= \frac{\Omega_{f_1} \cdot \frac{\iota_{2,4,4}}{\iota_{4,4,2}}}{\Omega_{h_3} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_1} \otimes \omega_{g_1} \otimes \omega_{h_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{f_1} \otimes \omega_{g_1})} \\ &= 15 \bmod 11^{39}. \end{aligned} \tag{7.4}$$

Meanwhile, we can easily check that

$$\frac{t_{4,6,8}!}{t_{8,4,6}!} \cdot \frac{t_{2,4,4}!}{t_{4,4,2}!} \cdot \frac{t_{8,8,2}!}{t_{2,8,8}!} = \frac{4! \cdot 2! \cdot 0!}{0! \cdot 0! \cdot 6!} = 1/15.$$

Therefore, we conclude that we need to take $\iota_{k,\ell,m} := 1/t_{k,\ell,m}!$ in Equation (7.1) and in the definition of $(f, g, h)_p$.

This makes sense, since we know that the author had mistakenly forgot to include the factor of $t!$ in Lemma 4.2.2 and Theorem 4.2.3, at the start of his research. Thankfully, it was possible to notice this mistake through Example 18, while the author was looking for other potential missing factors. Hence, thanks to this example, the author both realized that a term was missing in his initial formula, and gained confidence that no more coefficients would be needed in the definition of $(f, g, h)_p$.

It is only after concluding that the correct definition for $(f, g, h)_p$ was simply

$$(f, g, h)_p := \mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$$

that the author started investigating $\mathrm{AJ}_p(\Delta)(\omega_f \otimes \omega_g \otimes \omega_h)$ much more theoretically thus obtaining a proof for the desired symmetry result of Theorem 4.3.3 .

We now present one last example, in the same spirit as the previous one, that the author conducted as part of his investigation into potential missing factors in the definition of $(f, g, h)_p$. In it, we obtain the same conclusion as the previous example.

Example 19. Take $N = 42$ and $p = 11$. Let $f_0 \in S_2(\mathbb{Q}, 42)$, $f_1 \in S_4(\mathbb{Q}, 42)$, $g_1, g_2, g_3, g_4, g_5 \in S_6(\mathbb{Q}, 42)$ and $h_1, h_2, h_3, h_5 \in S_8(\mathbb{Q}, 42)$ be the cuspidal newforms given by

$$\begin{aligned} f_0 &= q + q^2 - q^3 + q^4 - 2q^5 - q^6 - q^7 + q^8 + q^9 - 2q^{10} - 4q^{11} + ..., \\ f_1 &= q + 2q^2 + 3q^3 + 4q^4 + 2q^5 + 6q^6 - 7q^7 + 8q^8 + 9q^9 + 4q^{10} - 8q^{11} + ..., \\ g_1 &= q + 4q^2 + 9q^3 + 16q^4 + 24q^5 + 36q^6 + 49q^7 + 64q^8 + 81q^9 + 96q^{10} + 66q^{11} + ..., \\ g_2 &= q + 4q^2 - 9q^3 + 16q^4 + 76q^5 - 36q^6 - 49q^7 + 64q^8 + 81q^9 + 304q^{10} + 650q^{11} + ..., \\ g_3 &= q - 4q^2 + 9q^3 + 16q^4 + 26q^5 - 36q^6 - 49q^7 - 64q^8 + 81q^9 - 104q^{10} + 664q^{11} + ..., \end{aligned}$$

$$g_4 = q - 4q^2 + 9q^3 + 16q^4 - 72q^5 - 36q^6 + 49q^7 - 64q^8 + 81q^9 + 288q^{10} - 414q^{11} + ...,$$
$$g_5 = q - 4q^2 - 9q^3 + 16q^4 + 44q^5 + 36q^6 - 49q^7 - 64q^8 + 81q^9 - 176q^{10} - 470q^{11} + ...,$$
$$h_1 = q + 8q^2 + 27q^3 + 64q^4 + 470q^5 + 216q^6 - 343q^7 + 512q^8 + 729q^9 + 3760q^{10} - 7268q^{11} + ...,$$
$$h_2 = q + 8q^2 - 27q^3 + 64q^4 + 30q^5 - 216q^6 + 343q^7 + 512q^8 + 729q^9 + 240q^{10} + 1788q^{11} + ...,$$
$$h_3 = q - 8q^2 + 27q^3 + 64q^4 - 122q^5 - 216 *^6 - 343q^7 - 512q^8 + 729q^9 + 976q^{10} - 1012q^{11} + ...,$$
$$h_5 = q - 8q^2 - 27q^3 + 64q^4 - 18q^5 + 216q^6 + 343q^7 - 512q^8 + 729q^9 + 144q^{10} + 8172q^{11} + ...$$

From Kedlaya's algorithm, we have

$$\Omega_{f_0} = -90348169492310771105171907193983150801 97 \cdot 11 \bmod 11^{40}.$$

Consider, separatly, the triples (f_0, g_1, g_4), (f_0, g_2, g_5), (f_0, h_1, h_3) and (f_0, h_2, h_5). We can compute

$$\begin{aligned}
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{g_1} \otimes \omega_{g_4}) &= 1246807517367640409859357102256181 55 \cdot 11^4 \bmod 11^{38}, \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_1} \otimes \omega_{g_4} \otimes \omega_{f_0}) &= -902519454720479784421987504992917 43 \cdot 11^2 \bmod 11^{36} \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{g_2} \otimes \omega_{g_5}) &= 48930540326539572005083693191318 57 \cdot 11^5 \bmod 11^{39}, \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_2} \otimes \omega_{g_5} \otimes \omega_{f_0}) &= -887391383513214522010199879189721 65 \cdot 11 \bmod 11^{35} \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_1} \otimes \omega_{h_3}) &= -4486165755767952572298931820600 57 \cdot 11^6 \bmod 11^{38}, \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{f_0} \otimes \omega_{h_1}) &= -7492585806757967492941728305606 66 \bmod 11^{32} \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_1} \otimes \omega_{h_3} \otimes \omega_{f_0}) &= -7842362214886483177364713073720 74 \cdot 11^{-1} \bmod 11^{31}, \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_2} \otimes \omega_{h_5}) &= -3469761359512472279550683621738 50 \cdot 11^7 \bmod 11^{39}, \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{h_5} \otimes \omega_{f_0}) &= -5342355255605669148628940966645 34 \bmod 11^{32}.
\end{aligned}$$

This allows us to recover the periods:

$$\begin{aligned}
\Omega_{g_1} \cdot \frac{\iota_{2,6,6}}{\iota_{6,6,2}} &= \Omega_{f_0} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{g_1} \otimes \omega_{g_4})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_1} \otimes \omega_{g_4} \otimes \omega_{f_0})} \\
&= 6040339681915204979406022697637356 8 \cdot 11^3 \bmod 11^{35} \\
\Omega_{g_2} \cdot \frac{\iota_{2,6,6}}{\iota_{6,6,2}} &= \Omega_{f_0} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{g_2} \otimes \omega_{g_5})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_2} \otimes \omega_{g_5} \otimes \omega_{f_0})} \\
&= 9027645613609416546767027822248692 8 \cdot 11^5 \bmod 11^{37} \\
\Omega_{h_3} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}} &= \Omega_{f_0} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_1} \otimes \omega_{h_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{h_1} \otimes \omega_{f_0})} \\
&= -526787756063977460369560891853684 \cdot 11^7 \bmod 11^{33} \\
\Omega_{h_1} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}} &= \Omega_{f_0} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_1} \otimes \omega_{h_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_1} \otimes \omega_{h_3} \otimes \omega_{f_0})} \\
&= -610255982304186836307627128142605 \cdot 11^8 \bmod 11^{34} \\
\Omega_{h_2} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}} &= \Omega_{f_0} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{f_0} \otimes \omega_{h_2} \otimes \omega_{h_5})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{h_5} \otimes \omega_{f_0})} \\
&= -694718239558038254512803761442819 \cdot 11^8 \bmod 11^{34}.
\end{aligned}$$

Consider now the triples (h_1, g_2, g_3), (h_2, g_1, g_3) and (g_2, f_1, h_3). We can compute

$$\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_1} \otimes \omega_{g_2} \otimes \omega_{g_3}) = 574938378583042257128547846305 27 \cdot 11 \bmod 11^{32},$$

$$
\begin{aligned}
\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_2} \otimes \omega_{g_3} \otimes \omega_{h_1}) &= 9219388492319049966191939698879 8 \cdot 11^4 \bmod 11^{35} \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{g_1} \otimes \omega_{g_3}) &= -2204588409664936079924992686773 0 \bmod 11^{31}, \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_1} \otimes \omega_{g_3} \otimes \omega_{h_2}) &= -9214361010454566333735603242879 2 \cdot 11^5 \bmod 11^{36} \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_2} \otimes \omega_{f_1} \otimes \omega_{h_3}) &= -1456642105829712076099530118300 \cdot 11^2 \bmod 11^{34}, \\
\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{g_2} \otimes \omega_{f_1}) &= -3178200652547504396392909208230 53 \bmod 11^{32}.
\end{aligned}
$$

Now, Equation (7.1) allow us to write:

$$
\begin{aligned}
\frac{\iota_{8,6,6}}{\iota_{6,6,8}} &= \frac{\Omega_{h_1}}{\Omega_{g_2}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_1} \otimes \omega_{g_2} \otimes \omega_{g_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_2} \otimes \omega_{g_3} \otimes \omega_{h_1})} \\
\frac{\iota_{8,6,6}}{\iota_{6,6,8}} &= \frac{\Omega_{h_2}}{\Omega_{g_1}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{g_1} \otimes \omega_{g_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_1} \otimes \omega_{g_3} \otimes \omega_{h_2})} \\
\frac{\iota_{6,4,8}}{\iota_{8,6,4}} &= \frac{\Omega_{g_2}}{\Omega_{h_3}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_2} \otimes \omega_{f_1} \otimes \omega_{h_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{g_2} \otimes \omega_{f_1})}.
\end{aligned}
$$

This allows us to finally obtain:

$$
\begin{aligned}
\frac{\iota_{8,6,6}}{\iota_{6,6,8}} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}} \cdot \frac{\iota_{6,6,2}}{\iota_{2,6,6}} &= \frac{\Omega_{h_1} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}}}{\Omega_{g_2} \cdot \frac{\iota_{2,6,6}}{\iota_{6,6,2}}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_1} \otimes \omega_{g_2} \otimes \omega_{g_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_2} \otimes \omega_{g_3} \otimes \omega_{h_1})} \\
&= 5^{-1} \bmod 11^{29} \\
\frac{\iota_{8,6,6}}{\iota_{6,6,8}} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}} \cdot \frac{\iota_{6,6,2}}{\iota_{2,6,6}} &= \frac{\Omega_{h_2} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}}}{\Omega_{g_1} \cdot \frac{\iota_{2,6,6}}{\iota_{6,6,2}}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_2} \otimes \omega_{g_1} \otimes \omega_{g_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_1} \otimes \omega_{g_3} \otimes \omega_{h_2})} \\
&= 5^{-1} \bmod 11^{30} \\
\frac{\iota_{6,4,8}}{\iota_{8,6,4}} \cdot \frac{\iota_{2,6,6}}{\iota_{6,6,2}} \cdot \frac{\iota_{8,8,2}}{\iota_{2,8,8}} &= \frac{\Omega_{g_2} \cdot \frac{\iota_{2,6,6}}{\iota_{6,6,2}}}{\Omega_{h_3} \cdot \frac{\iota_{2,8,8}}{\iota_{8,8,2}}} \cdot \frac{\mathrm{AJ}_p^\circ(\Delta)(\omega_{g_2} \otimes \omega_{f_1} \otimes \omega_{h_3})}{\mathrm{AJ}_p^\circ(\Delta)(\omega_{h_3} \otimes \omega_{g_2} \otimes \omega_{f_1})} \\
&= 15 \bmod 11^{31}.
\end{aligned}
\tag{7.5}
$$

Meanwhile, we can easily check that

$$
\begin{aligned}
\frac{t_{8,6,6}!}{t_{6,6,8}!} \cdot \frac{t_{2,8,8}!}{t_{8,8,2}!} \cdot \frac{t_{6,6,2}!}{t_{2,6,6}!} &= \frac{1! \cdot 6! \cdot 0!}{3! \cdot 0! \cdot 4!} = 5, \\
\frac{t_{6,4,8}!}{t_{8,6,4}!} \cdot \frac{t_{2,6,6}!}{t_{6,6,2}!} \cdot \frac{t_{8,8,2}!}{t_{2,8,8}!} &= \frac{2! \cdot 4! \cdot 0!}{0! \cdot 0! \cdot 6!} = 1/15.
\end{aligned}
$$

As expected, and as we have seen in Example 18, we need to take $\iota_{k,\ell,m} := 1/t_{k,\ell,m}!$ in Equation (7.1) and in the definition of $(f, g, h)_p$. This is consistent with Definition 4.2.4, given in Section 4.2, where we can see the factor $t!$ appearing in the formula of $(f, g, h)_p$.

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